Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices

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Free convolution with a semicircular distribution and eigenvalues of spiked deformations of Wigner matrices *

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Abstract

We investigate the asymptotic behavior of the eigenvalues of spiked perturbations of Wigner matrices defined by $M_N = \frac{1}{\sqrt{N}} W_N + A_N$, where W_N is a $N \times N$ Wigner Hermitian matrix whose entries have a distribution μ which is symmetric and satisfies a Poincaré inequality and A_N is a deterministic Hermitian matrix whose spectral measure converges to some probability measure ν with compact support. We assume that A_N has a fixed number of fixed eigenvalues (spikes) outside the support of ν whereas the distance between the other eigenvalues and the support of ν uniformly goes to zero as N goes to infinity. We establish that only a particular subset of the spikes will generate some eigenvalues of M_N which will converge to some limiting points outside the support of the limiting spectral measure. This phenomenon can be fully described in terms of free probability involving the subordination function related to the free additive convolution of ν by a semicircular distribution. Note that only finite rank perturbations had been considered up to now (even in the deformed GUE case).

Key words: Random matrices; Free probability; Deformed Wigner matrices; Asymptotic spectrum; Extreme eigenvalues; Stieltjes transform; Subordination property.

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1 Introduction

In the fifties, in order to describe the energy levels of a complex nuclei system by the eigenvalues of large Hermitian matrices, E. Wigner introduced the so-called Wigner $N \times N$ matrix W_N. According to Wigner's work [36], [37] and further results of different authors (see [3] for a review), provided the common distribution μ of the entries is centered with variance σ^2, the large N-limiting spectral distribution of the rescaled complex Wigner matrix $X_N = \frac{1}{\sqrt{N}} W_N$ is the semicircle distribution μ_σ whose density is given by

$$\frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2}\, \mathbf{1}_{[-2\sigma, 2\sigma]}(x). \tag{1.1}$$

Moreover, if the fourth moment of the measure μ is finite, the largest (resp. smallest) eigenvalue of X_N converges almost surely towards the right (resp. left) endpoint 2σ (resp. -2σ) of the semicircular support (cf. [7] or Theorem 2.12 in [3]).

Now, how does the spectrum behave under a deterministic Hermitian perturbation A_N? The set of possible spectra for $M_N = X_N + A_N$ depends in a complicated way on the spectra of X_N and A_N (see [21]). Nevertheless, when N becomes large, free probability provides us a good understanding of the global behavior of the spectrum of M_N. Indeed, if the spectral measure of A_N weakly converges to some probability measure ν and $\|A_N\|$ is uniformly bounded in N, the spectral distribution of M_N weakly converges to the free convolution $\mu_\sigma \boxplus \nu$ almost surely and in expectation (cf [1], [27] and [33], [19] for pioneering works). We refer the reader to [35] for an introduction to free probability theory. Note that when A_N is of finite rank, the spectral distribution of M_N still converges to the semicircular distribution ($\nu \equiv \delta_0$ and $\mu_\sigma \boxplus \nu = \mu_\sigma$).
In [30], S. Péché investigated the deformed GUE model $M_N^G = W_N^G/\sqrt{N} + A_N$, where W_N^G is a GUE matrix, that is a Wigner matrix associated to a centered Gaussian measure with variance σ^2 and A_N is a deterministic perturbation of finite rank with fixed eigenvalues. This model is the additive analogue of the Wishart matrices with spiked covariance matrix previously considered by J. Baik, G. Ben Arous and S. Péché [8] who exhibited a striking phase transition phenomenon for the fluctuations of the largest eigenvalue according to the values of the spikes. S. Péché pointed out an analogous phase transition phenomenon for the fluctuations of the largest eigenvalue of M_N^G with respect to the largest eigenvalue θ of A_N [30]. These investigations imply that, if θ is far enough from zero ($\theta > \sigma$), then the largest eigenvalue of M_N^G jumps above the support $[-2\sigma, 2\sigma]$ of the limiting spectral measure and converges (in probability) towards $\rho_\theta = \theta + \frac{\sigma^2}{\theta}$. Note that Z. Füredi and J. Komlós already exhibited such a phenomenon in [22] dealing with non-centered symmetric matrices.
In [20], D. Féral and S. Péché proved that the results of [30] still hold for a non-necessarily Gaussian Wigner Hermitian matrix W_N with sub-Gaussian moments and in the particular case of a rank one perturbation matrix A_N whose entries are all $\frac{\theta}{N}$ for some real number θ. In [18], we considered a deterministic Hermitian matrix A_N of arbitrary fixed finite rank r and built from a family of

J fixed non-null real numbers $\theta_1 > \cdots > \theta_J$ independent of N and such that each θ_j is an eigenvalue of A_N of fixed multiplicity k_j (with $\sum_{j=1}^J k_j = r$). In the following, the θ_j's are referred as the spikes of A_N. We dealt with general Wigner matrices associated to some symmetric measure satisfying a Poincaré inequality. We proved that eigenvalues of A_N with absolute value strictly greater than σ generate some eigenvalues of M_N which converge to some limiting points outside the support of μ_σ. To be more precise, we need to introduce further notations. Given an arbitrary Hermitian matrix B of size N, we denote by $\lambda_1(B) \geq \cdots \geq \lambda_N(B)$ its N ordered eigenvalues. For each spike θ_j, we denote by $n_{j-1}+1, \ldots, n_{j-1}+k_j$ the descending ranks of θ_j among the eigenvalues of A_N (multiplicities of eigenvalues are counted) with the convention that $k_1 + \cdots + k_{j-1} = 0$ for $j = 1$. One has that

$$n_{j-1} = k_1 + \cdots + k_{j-1} \quad \text{if } \theta_j > 0 \quad \text{and} \quad n_{j-1} = N - r + k_1 + \cdots + k_{j-1} \quad \text{if } \theta_j < 0.$$

Letting $J_{+\sigma}$ (resp. $J_{-\sigma}$) be the number of j's such that $\theta_j > \sigma$ (resp. $\theta_j < -\sigma$), we established in [18] that, when N goes to infinity,

a) for all j such that $1 \leq j \leq J_{+\sigma}$ (resp. $j \geq J - J_{-\sigma} + 1$), the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N), 1 \leq i \leq k_j)$ converge almost surely to $\rho_{\theta_j} = \theta_j + \frac{\sigma^2}{\theta_j}$ which is $> 2\sigma$ (resp. $< -2\sigma$).

b) $\lambda_{k_1+\cdots+k_{J_{+\sigma}}+1}(M_N) \xrightarrow{a.s.} 2\sigma$ and $\lambda_{N-(k_J+\cdots+k_{J-J_{-\sigma}+1})}(M_N) \xrightarrow{a.s.} -2\sigma$.

Actually, this phenomenon may be described in terms of free probability involving the subordination function related to the free convolution of $\nu = \delta_0$ by a semicircular distribution. Let us present it briefly. For a probability measure τ on $\mathbb{R}$, let us denote by g_τ its Stieltjes transform, defined for $z \in \mathbb{C} \setminus \mathbb{R}$ by

$$g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z - x}.$$

Let ν and τ be two probability measures on $\mathbb{R}$. It is proved in [13] Theorem 3.1 that there exists an analytic map $F : \mathbb{C}^+ \to \mathbb{C}^+$, called subordination function, such that

$$\forall z \in \mathbb{C}^+, g_{\tau \boxplus \nu}(z) = g_\nu(F(z)),$$

where $\mathbb{C}^+$ denotes the set of complex numbers z such that $\Im z > 0$. When $\tau = \mu_\sigma$, let us denote by $F_{\sigma,\nu}$ the corresponding subordination function. When $\nu = \delta_0$ and $\tau = \mu_\sigma$, the subordination function is given by $F_{\sigma,\delta_0} = 1/g_{\mu_\sigma}$. According to Lemma 4.4 in [18], one may notice that the complement of the support of $\mu_\sigma \boxplus \delta_0 (= \mu_\sigma)$ can be described as:

$$\mathbb{R} \setminus [-2\sigma, 2\sigma] = \{x, \exists u \in \mathbb{R}^*, |u| > \sigma \ \text{ such that } \ x = H_{\sigma,\delta_0}(u)\},$$

where $H_{\sigma,\delta_0}(z) = z + \frac{\sigma^2}{z}$ is the inverse function of the subordination function F_{σ,δ_0} on $\mathbb{R} \setminus [-2\sigma, 2\sigma]$. Now, the characterization of the spikes of A_N that generate jumps of eigenvalues of M_N i.e. $|\theta_j| > \sigma$ is obviously equivalent to the following

$$\theta_j \in \mathbb{R} \setminus \mathrm{supp}(\delta_0)(= \mathbb{R}^*) \quad \text{and} \quad H'_{\sigma,\delta_0}(\theta_j) > 0.$$

Moreover the relationship between a spike θ_j of A_N such that $|\theta_j| > \sigma$ and the limiting point ρ_{θ_j} of the corresponding eigenvalues of M_N (which is then outside $[-2\sigma; 2\sigma]$) is actually described by the inverse function of the subordination function as:

$$\rho_{\theta_j} = H_{\sigma,\delta_0}(\theta_j).$$

Actually this very interpretation in terms of subordination function of the characterization of the spikes of A_N that generate jumps of eigenvalues of M_N as well as the values of the jumps provides the intuition to imagine the generalization of the phenomenon dealing with non-finite rank perturbations just by replacing δ_0 by the limiting spectral distribution ν of A_N in the previous lines. Up to now, no result has been established for non-finite rank additive spiked perturbation. Moreover, this paper shows up that free probability can also shed light on the asymptotic behavior of the eigenvalues of the deformed Wigner model and strengthens the fact that free probability theory and random matrix theory are closely related.

More precisely, in this paper, we consider the following general deformed Wigner models $M_N = X_N + A_N$ such that:

- $X_N = \frac{1}{\sqrt{N}} W_N$ where W_N is a $N \times N$ Wigner Hermitian matrix associated to a distribution μ of variance σ^2 and mean zero: $(W_N)_{ii}$, $\sqrt{2}\Re((W_N)_{ij})_{i<j}$, $\sqrt{2}\Im((W_N)_{ij})_{i<j}$ are i.i.d., with distribution μ which is symmetric and satisfies a Poincaré inequality (the definition of such an inequality is recalled in the Appendix).

- A_N is a deterministic Hermitian matrix whose eigenvalues $\gamma_i^{(N)}$, denoted for simplicity by γ_i, are such that the spectral measure $\mu_{A_N} := \frac{1}{N}\sum_{i=1}^N \delta_{\gamma_i}$ converges to some probability measure ν with compact support. We assume that there exists a fixed integer $r \geq 0$ (independent from N) such that A_N has $N - r$ eigenvalues $\beta_j(N)$ satisfying

$$\max_{1 \leq j \leq N-r} \mathrm{dist}(\beta_j(N), \mathrm{supp}(\nu)) \underset{N\to\infty}{\longrightarrow} 0,$$

  where $\mathrm{supp}(\nu)$ denotes the support of ν. We also assume that there are J fixed real numbers $\theta_1 > \ldots > \theta_J$ independent of N which are outside the support of ν and such that each θ_j is an eigenvalue of A_N with a fixed multiplicity k_j (with $\sum_{j=1}^J k_j = r$). The θ_j's will be called the spikes or the spiked eigenvalues of A_N.

According to [1], the spectral distribution of M_N weakly converges to the free convolution $\mu_\sigma \boxplus \nu$ almost surely (cf. Remark 4.1 below). It turns out that the spikes of A_N that will generate jumps of eigenvalues of M_N will be the θ_j's such that $H'_{\sigma,\nu}(\theta_j) > 0$ where $H_{\sigma,\nu}(z) = z + \sigma^2 g_\nu(z)$ and the corresponding limiting points outside the support of $\mu_\sigma \boxplus \nu$ will be given by

$$\rho_{\theta_j} = H_{\sigma,\nu}(\theta_j).$$

It is worth noticing that the set $\{u \in \mathbb{R} \setminus \operatorname{supp}(\nu),\, H'_{\sigma,\nu}(u) > 0\}$ is actually the complement of the closure of the open set

$$U_{\sigma,\nu} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$$

introduced by P. Biane in [12] to describe the support of the free additive convolution of a probability measure ν on $\mathbb{R}$ by a semicircular distribution. Note that the deep study by P. Biane of the free convolution by a semicircular distribution will be of fundamental use in our approach. In Theorem 8.1, which is the main result of the paper, we present a complete description of the convergence of the eigenvalues of M_N depending on the location of the θ_j's with respect to $\overline{U_{\sigma,\nu}}$ and to the connected components of the support of ν.

Our approach also allows us to study the "non-spiked" deformed Wigner models i.e. such that $r = 0$. Up to now, the results which can be found in the literature for such a situation concern the so-called Gaussian matrix models with external source where the underlying Wigner matrix is from the GUE. Many works on these models deal with the local behavior of the eigenvalues of M_N (see for instance [14], [2] and [15] for details). Moreover, the recent results of [26] (which investigate several matrices in a free probability context) imply that the operator norm (i.e. the largest singular value) of some non-spiked deformed GUE $M_N^G = W_N^G/N + A_N$ converges almost surely to the L^∞-norm of a $(\mu_\sigma \boxplus \nu)$-distributed random variable. Here, we readily deduce (cf. Proposition 8.1 below) from our results the almost sure convergence of the extremal eigenvalues of general non-spiked deformed Wigner models to the corresponding endpoints of the compact support of the free convolution $\mu_\sigma \boxplus \nu$.

The asymptotic behavior of the eigenvalues of the deformed Wigner model M_N actually comes from two phenomena involving free convolution:

1. the inclusion of the spectrum of M_N in an ϵ-neighborhood of the support of $\mu_\sigma \boxplus \mu_{A_N}$, for all large N almost surely;

2. an exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, involving the subordination function $F_{\sigma,\nu}$ of $\mu_\sigma \boxplus \nu$ (i.e. to a gap in the spectrum of M_N, it corresponds through $F_{\sigma,\nu}$ a gap in the spectrum of A_N which splits the spectrum of A_N exactly as that of M_N).

The key idea to prove the first point is to obtain a precise estimate of order $\frac{1}{N}$ of the difference between the respective Stieltjes transforms of the mean spectral measure of the deformed model and of $\mu_\sigma \boxplus \mu_{A_N}$. To get such an estimate, we prove an "approximative subordination equation" satisfied by the Stieltjes transform of the deformed model. Note that, even if the ideas and tools are very close to those developed in [18], the proof in [18] does not use the above analysis from free probability whereas this very analysis allows us to extend the results of [18] to non-finite rank deformations. In particular, we didn't consider in [18] $\mu_\sigma \boxplus \mu_{A_N}$ whose support actually makes the asymptotic values of the eigenvalues that will be outside the limiting support of the spectral measure of M_N appear.

Note that phenomena 1. and 2. are actually the additive analogues of those described in [4], [5] in the framework of spiked population models, even if the authors do not refer to free probability. In [9], the authors use the results of [4], [5] to establish the almost sure convergence of the eigenvalues generated by the spikes in a spiked population model where all but finitely many eigenvalues of the covariance matrix are equal to one. Thus, they generalize the pioneering result of [8] in the Gaussian setting. Recently, [28], [6] extended this theory to a generalized spiked population model where the base population covariance matrix is arbitrary. Our results are exactly the additive analogues of theirs. It is worth noticing that one may check that these results on spiked population models could also be fully described in terms of free probability involving the subordination function related to the free multiplicative convolution of ν by a Marchenko-Pastur distribution.
Moreover, the results of F. Benaych-Georges and R. R. Nadakuditi in [11] about the convergence of the extremal eigenvalues of a matrix $X_N + A_N$, A_N being a finite rank perturbation whereas X_N is a unitarily invariant matrix with some compactly supported limiting spectral distribution μ, could be rewritten in terms of the subordination function related to the free additive convolution of δ_0 by μ. Hence, we think that subordination property in free probability definitely sheds light on spiked deformed models.

Finally, one can expect that our results hold true in a more general setting than the one considered here, namely only requires the existence of a finite fourth moment on the measure μ of the Wigner entries. Nevertheless, the assumption that μ satisfies a Poincaré inequality is fundamental in our approach since we need several variance estimates.

The paper is organized as follows. In Section 2, we first recall some results on free additive convolution and subordination property as well as the description by P. Biane of the support of the free convolution of some probability measure ν by a semicircular distribution. We then deduce a characterization of this support via the subordination function when ν is compactly supported and we exhibit relationships between the steps of the distribution functions of ν and $\mu_\sigma \boxplus \nu$. In Section 3, we establish an approximative subordination equation for the Stieltjes transform g_N of the mean spectral distribution of the deformed model M_N and explain in Section 4 how to deduce an estimation up to the order $\frac{1}{N^2}$ of the difference between g_N and the Stieltjes transform of $\mu_\sigma \boxplus \mu_{A_N}$ when N goes to infinity. In Section 5, we show how to deduce the almost sure inclusion of the spectrum of M_N in a neighborhood of the support of $\mu_\sigma \boxplus \mu_{A_N}$ for all large N; we use the ideas (based on inverse Stieltjes tranform) of [23] and [31] in the non-deformed Gaussian complex, real or symplectic Wigner setting; nevertheless, since $\mu_\sigma \boxplus \mu_{A_N}$ depends on N, we need here to apply the inverse Stieltjes tranform to functions depending on N and we therefore give the details of the proof to convince the reader that the approach developped by [23] and [31] still holds. In Section 6, we show how the support of $\mu_\sigma \boxplus \mu_{A_N}$ makes the asymptotic values of the eigenvalues that will be outside the support of the limiting spectral measure appear since we prove that, for any $\epsilon > 0$, $\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})$ is included in an ϵ-neighborhood of $\mathrm{supp}(\mu_\sigma \boxplus \nu) \bigcup \{\rho_{\theta_j},\ \theta_j \text{ such that } H'_{\sigma,\nu}(\theta_j) > 0\}$, when N is large enough. Section 7 is devoted to the proof of the exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, involving the subordination function $F_{\sigma,\nu}$. In the last section, we show how to deduce our main result (Theorem 8.1) about the convergence of the eigenvalues of the deformed model M_N. Finally we present in an Appendix the proofs of some technical estimates on variances used throughout the paper.

Throughout this paper, we will use the following notations.

- For a probability measure τ on $\mathbb{R}$, we denote by g_τ its Stieltjes transform defined for $z \in \mathbb{C} \setminus \mathbb{R}$ by

$$g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z - x}.$$

- G_N denotes the resolvent of M_N and g_N the mean of the Stieltjes transform of the spectral measure of M_N, that is

$$g_N(z) = \mathbb{E}(\mathrm{tr}_N G_N(z)),\ z \in \mathbb{C} \setminus \mathbb{R},$$

  where tr_N is the normalized trace: $\mathrm{tr}_N = \frac{1}{N}\mathrm{Tr}$.
  We recall some useful properties of the resolvent (see [25], [17]).

  Lemma 1.1. *For a $N \times N$ Hermitian or symmetric matrix M, for any $z \in \mathbb{C} \setminus \mathrm{Spect}(M)$, we denote by $G(z) := (zI_N - M)^{-1}$ the resolvent of M. Let $z \in \mathbb{C} \setminus \mathbb{R}$,*

  (i) $\|G(z)\| \leq |\Im z|^{-1}$ *where* $\|.\|$ *denotes the operator norm.*

  (ii) $|G(z)_{ij}| \leq |\Im z|^{-1}$ *for all* $i, j = 1, \ldots, N$.

  (iii) For $p \geq 2$,
$$\frac{1}{N} \sum_{i,j=1}^{N} |G(z)_{ij}|^p \leq (|\Im z|^{-1})^p. \tag{1.2}$$

  (iv) The derivative with respect to M of the resolvent $G(z)$ satisfies:
$$G'_M(z).B = G(z)BG(z) \quad \textit{for any matrix } B.$$

  (v) Let $z \in \mathbb{C}$ such that $|z| > \|M\|$; we have
$$\|G(z)\| \leq \frac{1}{|z| - \|M\|}.$$

- $\tilde{g}_N$ denotes the Stieltjes transform of the probability measure $\mu_\sigma \boxplus \mu_{A_N}$.

- When we state that some quantity $\Delta_N(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, is $O(\frac{1}{N^p})$, this means precisely that:
$$|\Delta_N(z)| \leq \frac{P(|\Im z|^{-1})}{N^p},$$
for some polynomial P with nonnegative coefficients which is independent of N.

- For any set S in $\mathbb{R}$, we denote the set $\{x \in \mathbb{R}, \mathrm{dist}(x,S) \leq \epsilon\}$ (resp. $\{x \in \mathbb{R}, \mathrm{dist}(x,S) < \epsilon\}$) by $S + [-\epsilon, +\epsilon]$ (resp. $S + (-\epsilon, +\epsilon)$).

2 Free convolution

2.1 Definition and subordination property

Let τ be a probability measure on $\mathbb{R}$. Its Stieltjes transform g_τ is analytic on the complex upper half-plane $\mathbb{C}^+$. There exists a domain
$$D_{\alpha,\beta} = \{u + iv \in \mathbb{C}, |u| < \alpha v, v > \beta\}$$
on which g_τ is univalent. Let K_τ be its inverse function, defined on $g_\tau(D_{\alpha,\beta})$, and
$$R_\tau(z) = K_\tau(z) - \frac{1}{z}.$$
Given two probability measures τ and ν, there exists a unique probability measure λ such that
$$R_\lambda = R_\tau + R_\nu$$
on a domain where these functions are defined. The probability measure λ is called the free convolution of τ and ν and denoted by $\tau \boxplus \nu$.

The free convolution of probability measures has an important property, called subordination, which can be stated as follows: let τ and ν be two probability measures on $\mathbb{R}$; there exists an analytic map $F : \mathbb{C}^+ \to \mathbb{C}^+$ such that
$$\forall z \in \mathbb{C}^+, \quad g_{\tau\boxplus\nu}(z) = g_\nu(F(z)).$$
This phenomenon was first observed by D. Voiculescu under a genericity assumption in [34], and then proved in generality in [13] Theorem 3.1. Later, a new proof of this result was given in [10], using a fixed point theorem for analytic self-maps of the upper half-plane.

2.2 Free convolution by a semicircular distribution

In [12], P. Biane provides a deep study of the free convolution by a semicircular distribution. We first recall here some of his results that will be useful in our approach.

Let ν be a probability measure on $\mathbb{R}$. P. Biane [12] introduces the set
$$\Omega_{\sigma,\nu} := \{u + iv \in \mathbb{C}^+, v > v_{\sigma,\nu}(u)\},$$

where the function $v_{\sigma,\nu} : \mathbb{R} \to \mathbb{R}^+$ is defined by

$$v_{\sigma,\nu}(u) = \inf \left\{ v \geq 0, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} \leq \frac{1}{\sigma^2} \right\}$$

and proves the following

Proposition 2.1. *[12] The map*

$$H_{\sigma,\nu} : z \longmapsto z + \sigma^2 g_\nu(z)$$

is a homeomorphism from $\overline{\Omega_{\sigma,\nu}}$ to $\mathbb{C}^+ \cup \mathbb{R}$ which is conformal from $\Omega_{\sigma,\nu}$ onto $\mathbb{C}^+$. Let $F_{\sigma,\nu} : \mathbb{C}^+ \cup \mathbb{R} \to \overline{\Omega_{\sigma,\nu}}$ be the inverse function of $H_{\sigma,\nu}$. One has,

$$\forall z \in \mathbb{C}^+, \quad g_{\mu_\sigma \boxplus \nu}(z) = g_\nu(F_{\sigma,\nu}(z))$$

and then

$$F_{\sigma,\nu}(z) = z - \sigma^2 g_{\mu_\sigma \boxplus \nu}(z). \tag{2.1}$$

Note that in particular the Stieltjes transform $\tilde{g}_N$ of $\mu_\sigma \boxplus \mu_{A_N}$ satisfies

$$\forall z \in \mathbb{C}^+, \quad \tilde{g}_N(z) = g_{\mu_{A_N}}(z - \sigma^2 \tilde{g}_N(z)). \tag{2.2}$$

Considering $H_{\sigma,\nu}$ as an analytic map defined in the whole upper half-plane $\mathbb{C}^+$, it is clear that

$$\Omega_{\sigma,\nu} = H_{\sigma,\nu}^{-1}(\mathbb{C}^+). \tag{2.3}$$

Let us give a quick proof of (2.3). Let $v > 0$. Since

$$\Im H_{\sigma,\nu}(u + iv) = v\left(1 - \sigma^2 \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2}\right),$$

we have

$$\Im H_{\sigma,\nu}(u + iv) > 0 \Longleftrightarrow \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} < \frac{1}{\sigma^2}. \tag{2.4}$$

Consequently one can easily see that $\Omega_{\sigma,\nu}$ is included in $H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$. Moreover if $u + iv \in H_{\sigma,\nu}^{-1}(\mathbb{C}^+)$ then (2.4) implies that $v \geq v_{\sigma,\nu}(u)$. If we assume that $v = v_{\sigma,\nu}(u)$, then $v_{\sigma,\nu}(u) > 0$ and finally

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} = \frac{1}{\sigma^2}$$

by Lemma 2 in [12]. This is a contradiction : necessarily $v > v_{\sigma,\nu}(u)$ or, in other words, $u + iv \in \Omega_{\sigma,\nu}$ and we are done.

The previous results of P. Biane allow him to conclude that $\mu_\sigma \boxplus \nu$ is absolutely continuous with respect to the Lebesgue measure and to obtain the following description of the support.

Theorem 2.1. *[12] Define $\Psi_{\sigma,\nu} : \mathbb{R} \to \mathbb{R}$ by:*

$$\Psi_{\sigma,\nu}(u) = H_{\sigma,\nu}(u + iv_{\sigma,\nu}(u)) = u + \sigma^2 \int_{\mathbb{R}} \frac{(u-x)d\nu(x)}{(u-x)^2 + v_\sigma(u)^2}.$$

$\Psi_{\sigma,\nu}$ is a homeomorphism and, at the point $\Psi_{\sigma,\nu}(u)$, the measure $\mu_\sigma \boxplus \nu$ has a density given by

$$p_{\sigma,\nu}(\Psi_{\sigma,\nu}(u)) = \frac{v_{\sigma,\nu}(u)}{\pi\sigma^2}.$$

Define the set

$$U_{\sigma,\nu} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\} = \{u \in \mathbb{R}, v_{\sigma,\nu}(u) > 0\}.$$

The support of the measure $\mu_\sigma \boxplus \nu$ is the image of the closure of the open set $U_{\sigma,\nu}$ by the homeomorphism $\Psi_{\sigma,\nu}$. $\Psi_{\sigma,\nu}$ is strictly increasing on $U_{\sigma,\nu}$.

Hence,

$$\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu) = \Psi_{\sigma,\nu}(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}).$$

One has $\Psi_{\sigma,\nu} = H_{\sigma,\nu}$ on $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ and $\Psi_{\sigma,\nu}^{-1} = F_{\sigma,\nu}$ on $\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu)$. In particular, we have the following description of the complement of the support:

$$\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu) = H_{\sigma,\nu}(\mathbb{R} \setminus \overline{U_{\sigma,\nu}}). \tag{2.5}$$

Let ν be a compactly supported probability measure. We are going to establish a characterization of the complement of the support of $\mu_\sigma \boxplus \nu$ involving the support of ν and $H_{\sigma,\nu}$. We will need the following preliminary lemma.

Lemma 2.1. *The support of ν is included in $\overline{U_{\sigma,\nu}}$.*

Proof of Lemma 2.1: Let x_0 be in $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$. Then, there is some $\epsilon > 0$ such that $[x_0 - \epsilon, x_0 + \epsilon] \subset \mathbb{R} \setminus \overline{U_{\sigma,\nu}}$. For any integer $n \geq 1$, we define $\alpha_k = x_0 - \epsilon + 2k\epsilon/n$ for all $0 \leq k \leq n$. Then, as the sets $[\alpha_k, \alpha_{k+1}]$ are trivially contained in $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$, one has that:

$$\forall u \in [\alpha_k, \alpha_{k+1}], \quad \frac{1}{\sigma^2} \geq \int_{\alpha_k}^{\alpha_{k+1}} \frac{d\nu(x)}{(u-x)^2} \geq \frac{\nu([\alpha_k, \alpha_{k+1}])}{(\alpha_{k+1} - \alpha_k)^2}.$$

This readily implies that

$$\nu([x_0 - \epsilon, x_0 + \epsilon]) \leq \sum_{k=0}^{n-1} \nu([\alpha_k, \alpha_{k+1}]) \leq \frac{(2\epsilon)^2}{\sigma^2 n}.$$

Letting $n \to \infty$, we get that $\nu([x_0 - \epsilon, x_0 + \epsilon]) = 0$, which implies that $x_0 \in \mathbb{R} \setminus \operatorname{supp}(\nu)$. $\square$

From the continuity and strict convexity of the function $u \longrightarrow \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2}$ on $\mathbb{R} \setminus \operatorname{supp}(\nu)$, it follows that

$$\overline{U_{\sigma,\nu}} = \operatorname{supp}(\nu) \cup \{u \in \mathbb{R} \setminus \operatorname{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} \geq \frac{1}{\sigma^2}\} \tag{2.6}$$

and

$$\mathbb{R} \setminus \overline{U_{\sigma,\nu}} = \{u \in \mathbb{R} \setminus \text{supp}(\nu), \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2}\}.$$

Now, as $H_{\sigma,\nu}$ is analytic on $\mathbb{R} \setminus \text{supp}(\nu)$, the following characterization readily follows:

$$\mathbb{R} \setminus \overline{U_{\sigma,\nu}} = \{u \in \mathbb{R} \setminus \text{supp}(\nu),\, H'_{\sigma,\nu}(u) > 0\}.$$

and thus, according to (2.5), we get

Proposition 2.2.

$x \in \mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \nu) \Leftrightarrow \exists u \in \mathbb{R} \setminus \text{supp}(\nu)$ such that $x = H_{\sigma,\nu}(u)\,,\, H'_{\sigma,\nu}(u) > 0.$

Remark 2.1. *Note that $H_{\sigma,\nu}$ is strictly increasing on $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ since, if $a < b$ are in $\mathbb{R} \setminus \text{supp}(\nu)$, one has, by Cauchy-Schwarz inequality, that*

$$\begin{aligned} H_{\sigma,\nu}(b) - H_{\sigma,\nu}(a) &= (b-a)\left[1 - \sigma^2 \int_{\mathbb{R}} \frac{d\nu(x)}{(a-x)(b-x)}\right] \\ &\geq (b-a)\left[1 - \sigma^2 \sqrt{(-g'_\nu(a))(-g'_\nu(b))}\right]. \end{aligned}$$

which is nonnegative if a and b belong to $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$.

Remark 2.2. *Each connected component of $\overline{U_{\sigma,\nu}}$ contains at least one connected component of* $\text{supp}(\nu)$*.*

Indeed, let $[s_l, t_l]$ be a connected component of $\overline{U_{\sigma,\nu}}$. If s_l or t_l is in $\text{supp}(\nu)$, $[s_l, t_l]$ contains at least a connected component of $\text{supp}(\nu)$ since $\text{supp}(\nu)$ is included in $\overline{U_{\sigma,\nu}}$. Now, if neither s_l nor t_l is in $\text{supp}(\nu)$, according to (2.6), we have

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(s_l-x)^2} = \int_{\mathbb{R}} \frac{d\nu(x)}{(t_l-x)^2} = \frac{1}{\sigma^2}.$$

Assume that $[s_l, t_l] \subset \mathbb{R} \setminus \text{supp}(\nu)$, then, by strict convexity of the function $u \longmapsto \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2}$ on $\mathbb{R} \setminus \text{supp}(\nu)$, one obtains that, for any $u \in]s_l, t_l[$,

$$\int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} < \frac{1}{\sigma^2},$$

which leads to a contradiction. $\Box$

Remark 2.3. *One can readily see that*

$$\overline{U_{\sigma,\nu}} \subset \{u, \text{dist}(u, \text{supp}(\nu)) \leq \sigma\}$$

and deduce, since $\text{supp}(\nu)$ *is compact, that $U_{\sigma,\nu}$ is a relatively compact open set. Hence, $\overline{U_{\sigma,\nu}}$ has a finite number of connected components and may be written as the following finite disjoint union*

$$\overline{U_{\sigma,\nu}} = \bigcup_{l=m}^{1} [s_l, t_l] \quad \textit{with } s_m < t_m < \ldots < s_1 < t_1. \tag{2.7}$$

We close this section with a proposition pointing out a relationship between the distribution functions of ν and $\mu_\sigma \boxplus \nu$.

Proposition 2.3. *Let $[s_l, t_l]$ be a connected component of $\overline{U_{\sigma,\nu}}$, then*

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(s_l), \Psi_{\sigma,\nu}(t_l)]) = \nu([s_l, t_l]).$$

Proof of Proposition 2.3: Let $]a, b[$ be a connected component of $U_{\sigma,\nu}$. Since a and b are not atoms of ν and $\mu_\sigma \boxplus \nu$ is absolutely continuous, it is enough to show

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(a), \Psi_{\sigma,\nu}(b)]) = \nu([a, b]).$$

From Cauchy's inversion formula, $\mu_\sigma \boxplus \nu$ has a density given by $p_\sigma(x) = -\frac{1}{\pi}\Im(g_\nu(F_{\nu,\sigma}(x))$ and

$$(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(a), \Psi_{\sigma,\nu}(b)]) = -\frac{1}{\pi}\Im\left(\int_{\Psi_{\sigma,\nu}(a)}^{\Psi_{\sigma,\nu}(b)} g_\nu(F_{\nu,\sigma}(x))dx\right).$$

We set $z = F_{\sigma,\nu}(x)$, then $x = H_{\sigma,\nu}(z)$ and $z = u + iv_{\sigma,\nu}(u)$. Note that $v_{\sigma,\nu}(u) > 0$ for $u \in]a, b[$ and $v_{\sigma,\nu}(a) = v_{\sigma,\nu}(b) = 0$ (see [12]). Then,

$$\begin{aligned}
&(\mu_\sigma \boxplus \nu)([\Psi_{\sigma,\nu}(a), \Psi_{\sigma,\nu}(b)]) \\
&= -\frac{1}{\pi}\Im\left(\int_a^b g_\nu(u + iv_{\sigma,\nu}(u))H'_{\sigma,\nu}(u + iv_{\sigma,\nu}(u))(1 + iv'_{\sigma,\nu}(u))du\right) \\
&= -\frac{1}{\pi}\Im\left(\int_a^b g_\nu(u + iv_{\sigma,\nu}(u))(1 + \sigma^2 g'_\nu(u + iv_{\sigma,\nu}(u)))(1 + iv'_{\sigma,\nu}(u))du\right) \\
&= -\frac{1}{\pi}\left(\Im\int_a^b g_\nu(u + iv_{\sigma,\nu}(u))(1 + iv'_{\sigma,\nu}(u))du + \frac{\sigma^2}{2}\Im[g_\nu^2(u + iv_{\sigma,\nu}(u))]_a^b\right) \\
&= -\frac{1}{\pi}\Im\int_a^b g_\nu(u + iv_{\sigma,\nu}(u))(1 + iv'_{\sigma,\nu}(u))du = -\frac{1}{\pi}\Im\int_\gamma g_\nu(z)dz,
\end{aligned}$$

where

$$\gamma = \{z = u + iv_{\sigma,\nu}(u), u \in [a, b]\}.$$

Now, we recall that, since a and b are points of continuity of the distribution function of ν,

$$\nu([a, b]) = \lim_{\epsilon\to 0} -\frac{1}{\pi}\Im\left(\int_a^b g_\nu(u + i\epsilon)du\right) = \lim_{\epsilon\to 0} -\frac{1}{\pi}\Im\left(\int_{\gamma_\epsilon} g_\nu(z)dz\right),$$

where $\gamma_\epsilon = \{z = u + i\epsilon, u \in [a, b]\}$. Thus, it remains to prove that:

$$\lim_{\epsilon\to 0}\left(\Im\left(\int_\gamma g_\nu(z)dz\right) - \Im\left(\int_{\gamma_\epsilon} g_\nu(z)dz\right)\right) = 0. \tag{2.8}$$

Let $\epsilon > 0$ such that $\epsilon < \sup_{[a,b]} v_{\sigma,\nu}(u)$. We introduce the contour

$$\hat{\gamma}_\epsilon = \{z = u + i(v_{\sigma,\nu}(u) \wedge \epsilon), u \in [a,b]\}.$$

From the analyticity of g_ν on $\mathbb{C}^+$, we have

$$\int_\gamma g_\nu(z)dz = \int_{\hat{\gamma}_\epsilon} g_\nu(z)dz.$$

Let $I_\epsilon = \{u \in [a,b], v_{\sigma,\nu}(u) < \epsilon\} = \cup C_i(\epsilon)$, where $C_i(\epsilon)$ are the connected components of I_ϵ. Then, $I_\epsilon \downarrow_{\epsilon\to 0} \{a,b\}$. For $u \in I_\epsilon$,

$$|\Im g_\nu(u+i\epsilon)| = \epsilon \int \frac{d\nu(x)}{(u-x)^2+\epsilon^2} \leq \epsilon \int \frac{d\nu(x)}{(u-x)^2 + v_{\sigma,\nu}^2(u)} \leq \frac{\epsilon}{\sigma^2}$$

and

$$\int_{I_\epsilon} |\Im g_\nu(u+i\epsilon)|du \leq \frac{\epsilon}{\sigma^2}(b-a).$$

On the other hand, for $u \in I_\epsilon$,

$$|\Im g_\nu(u + iv_{\sigma,\nu}(u))| = v_{\sigma,\nu}(u) \int \frac{d\nu(x)}{(u-x)^2 + v_{\sigma,\nu}(u)^2} \leq \frac{\epsilon}{\sigma^2}.$$

Moreover,

$$\Re g_\nu(u + iv_{\sigma,\nu}(u))v'_{\sigma,\nu}(u) = \frac{\Psi_{\sigma,\nu}(u) - u}{\sigma^2} v'_{\sigma,\nu}(u)$$

and

$$\begin{aligned}
\int_{I_\epsilon} \Re g_\nu(u + iv_{\sigma,\nu}(u))v'_{\sigma,\nu}(u)du &= \int_{I_\epsilon} \frac{\Psi_{\sigma,\nu}(u) - u}{\sigma^2} v'_{\sigma,\nu}(u)du \\
&= \frac{1}{\sigma^2}\sum_i [(\Psi_{\sigma,\nu}(u) - u)v_{\sigma,\nu}(u)]_{C_i(\epsilon)} \\
&\quad -\frac{1}{\sigma^2}\int_{I_\epsilon} (\Psi'_{\sigma,\nu}(u) - 1)v_{\sigma,\nu}(u)du,
\end{aligned}$$

by integration by parts. Now (see [12] or Theorem 2.1),

$$\int_{I_\epsilon} \Psi'_{\sigma,\nu}(u)v_{\sigma,\nu}(u)du = \pi\sigma^2(\mu_\sigma \boxplus \nu)(\Psi_{\sigma,\nu}(I_\epsilon)) \underset{\epsilon\to 0}{\longrightarrow} 0.$$

$$\int_{I_\epsilon} v_{\sigma,\nu}(u)du \leq \epsilon(b-a).$$

Since $\Psi_{\sigma,\nu}$ is increasing on $[a,b]$,

$$\sum_i [\Psi_{\sigma,\nu}(u)v_{\sigma,\nu}(u)]_{C_i(\epsilon)} \leq \epsilon(\Psi_{\sigma,\nu}(b) - \Psi_{\sigma,\nu}(a))$$

and

$$\sum_i [uv_{\sigma,\nu}(u)]_{C_i(\epsilon)} \leq \epsilon(b-a).$$

The above inequalities imply (2.8). $\square$

3 Approximate subordination equation for $g_N(z)$

We look for an approximative equation for $g_N(z)$ of the form (2.2). To estimate $g_N(z)$, we first handle the simplest case where W_N is a GUE matrix and then see how the equation is modified in the general Wigner case. We shall rely on an integration by parts formula. The first integration by parts formula concerns the Gaussian case; the distribution μ associated to W_N is a centered Gaussian distribution with variance σ^2 and the resulting distribution of $X_N = W_N/\sqrt{N}$ is denoted by GUE$(N, \sigma^2/N)$. Then, the integration by parts formula can be expressed in a matricial form.

Lemma 3.1. *Let Φ be a complex-valued $\mathcal{C}^1$ function on $(M_N(\mathbb{C})_{sa})$ and $X_N \sim$ GUE$(N, \frac{\sigma^2}{N})$. Then,*

$$\mathbb{E}[\Phi'(X_N).H] = \frac{N}{\sigma^2}\mathbb{E}[\Phi(X_N)\mathrm{Tr}(X_N H)], \tag{3.1}$$

for any Hermitian matrix H, or by linearity for $H = E_{jk}$, $1 \leq j, k \leq N$, where $(E_{jk})_{1\leq j,k\leq N}$ is the canonical basis of the complex space of $N \times N$ matrices.

For a general distribution μ, we shall use an "approximative" integration by parts formula, applied to the variable $\xi = \sqrt{2}\Re((X_N)_{kl})$ or $\sqrt{2}\Im((X_N)_{kl})$, $k < l$, or $(X_N)_{kk}$. Note that for $k < l$ the derivative of $\Phi(X_N)$ with respect to $\sqrt{2}\Re((X_N)_{kl})$ (resp. $\sqrt{2}\Im((X_N)_{kl})$) is $\Phi'(X_N).e_{kl}$ (resp. $\Phi'(X_N).f_{kl}$), where $e_{kl} = \frac{1}{\sqrt{2}}(E_{kl} + E_{lk})$ (resp. $f_{kl} = \frac{i}{\sqrt{2}}(E_{kl} - E_{lk})$) and for any k, the derivative of $\Phi(X_N)$ with respect to $(X_N)_{kk}$ is $\Phi'(X_N).E_{kk}$.

Lemma 3.2. *Let ξ be a real-valued random variable such that $\mathbb{E}(|\xi|^{p+2}) < \infty$. Let ϕ be a function from $\mathbb{R}$ to $\mathbb{C}$ such that the first $p+1$ derivatives are continuous and bounded. Then,*

$$\mathbb{E}(\xi\phi(\xi)) = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!}\mathbb{E}(\phi^{(a)}(\xi)) + \epsilon, \tag{3.2}$$

where κ_a are the cumulants of ξ, $|\epsilon| \leq C \sup_t |\phi^{(p+1)}(t)|\mathbb{E}(|\xi|^{p+2})$, C only depends on p.

Let $U(= U(N))$ be a unitary matrix such that

$$A_N = U^*\mathrm{diag}(\gamma_1, \ldots, \gamma_N)U$$

and let G stand for $G_N(z)$. Consider $\tilde{G} = UGU^*$. We describe the approach in the Gaussian case and present the corresponding results in the general Wigner case but detail some technical proofs in the Appendix.

a) Gaussian case: We apply (3.1) to $\Phi(X_N) = G_{jl}$, $H = E_{il}$, $1 \leq i, j, l \leq N$, and then take $\frac{1}{N}\sum_l$ to obtain, using the resolvent equation $GX_N = -I + zG - GA_N$ (see [18]),

$$Z_{ji} := \sigma^2\mathbb{E}[G_{ji}\mathrm{tr}_N(G)] + \delta_{ij} - z\mathbb{E}(G_{ji}) + \mathbb{E}[(GA_N)_{ji}] = 0.$$

Now, let $1 \leq k, p \leq N$ and consider the sum $\sum_{i,j} U_{ik}^* U_{pj} Z_{ji}$. We obtain from the previous equation

$$\sigma^2 \mathbb{E}[\tilde{G}_{pk} \mathrm{tr}_N(G)] + \delta_{pk} - z\mathbb{E}(\tilde{G}_{pk}) + \gamma_k \mathbb{E}[\tilde{G}_{pk}] = 0. \tag{3.3}$$

Hence, using Lemma 9.2 in the Appendix stating that

$$|\mathbb{E}[\tilde{G}_{pk} \mathrm{tr}_N(G)] - \mathbb{E}[\tilde{G}_{pk}]\mathbb{E}[\mathrm{tr}_N(G)]| = O(\frac{1}{N^2}),$$

we finally get the following estimation

$$\mathbb{E}(\tilde{G}_{pk}) = \frac{\delta_{pk}}{(z - \sigma^2 g_N(z) - \gamma_k)} + O(\frac{1}{N^2}), \tag{3.4}$$

where we use that $|\frac{1}{z-\sigma^2 g_N(z)-\gamma_i}| \leq |\Im z|^{-1}$, and then

$$\begin{aligned} g_N(z) &= \frac{1}{N}\sum_{k=1}^{N} \mathbb{E}[\tilde{G}_{kk}] = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z - \sigma^2 g_N(z) - \gamma_k} + O(\frac{1}{N^2}) \\ &= \int_{\mathbb{R}} \frac{1}{z - \sigma^2 g_N(z) - x} d\mu_{A_N}(x) + O(\frac{1}{N^2}) \\ &= g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + O(\frac{1}{N^2}). \end{aligned}$$

In the Gaussian case, we have thus proved:

Proposition 3.1. *For $z \in \mathbb{C}^+$, $g_N(z)$ satisfies:*

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + O\left(\frac{1}{N^2}\right). \tag{3.5}$$

b) Non-Gaussian case: In this case, the integration by parts formula gives the following generalization of (3.4):

Lemma 3.3.

$$\mathbb{E}(\tilde{G}_{pk}) = \frac{\delta_{pk}}{(z - \sigma^2 g_N(z) - \gamma_k)} + \frac{\kappa_4}{2N^2}\frac{\mathbb{E}[\tilde{A}(p,k)]}{(z - \sigma^2 g_N(z) - \gamma_k)} + O(\frac{1}{N^2}), \tag{3.6}$$

where

$$\begin{aligned} \tilde{A}(p,k) &= \sum_{i,j} U_{ik}^* U_{pj} \left\{ \sum_l G_{jl} G_{il}^3 + \sum_l G_{ji} G_{il} G_{li} G_{ll} \right. \\ &\quad \left. + \sum_l G_{jl} G_{ii} G_{li} G_{ll} + \sum_l G_{ji} G_{ii} G_{ll}^2 \right\} \end{aligned} \tag{3.7}$$

and $\frac{1}{N^2}\tilde{A}(p,k) \leq C\frac{|\Im z|^{-4}}{N}$.

Proof Lemma 3.3 readily follows from (9.4), Lemma 9.2 and (9.3) established in the Appendix. □

Thus,

$$\begin{aligned} g_N(z) &= \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}[\tilde{G}_{kk}] = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{z-\sigma^2 g_N(z)-\gamma_k} \\ &+ \frac{\kappa_4}{2N^3}\sum_{k=1}^{N}\frac{\mathbb{E}[\tilde{A}(k,k)]}{z-\sigma^2 g_N(z)-\gamma_k} + O(\frac{1}{N^2}). \end{aligned}$$

Let us show that the first three terms in $\frac{1}{N}\sum_k \mathbb{E}[\tilde{A}(k,k)]/(z-\sigma^2 g_N(z)-\gamma_k)$ coming from the decomposition (3.7) are bounded and thus give a $O(\frac{1}{N^2})$ contribution in $g_N(z)$. We denote by G_D the diagonal matrix with k-th diagonal entry equal to $\frac{1}{z-\sigma^2 g_N(z)-\gamma_k}$.

$$\begin{aligned} \left|\sum_{i,j,k} U^*_{ik}U_{kj}\frac{1}{z-\sigma^2 g_N(z)-\gamma_k}\mathbb{E}[\sum_l G_{jl}G^3_{il}]\right| &= \left|\mathbb{E}[\sum_{i,l}(U^*G_DUG)_{il}G^3_{il}]\right| \\ &\leq |\Im z|^{-2}\mathbb{E}[\sum_{i,l}|G^3_{il}|] \\ &\leq |\Im z|^{-5}N, \end{aligned}$$

using Lemma 1.1. The second term is of the same kind. For the third term, we obtain

$$|\sum_i (U^*G_DUG^2G^{(d)})_{ii}G_{ii}| \leq |\Im z|^{-5}N$$

where $G^{(d)}$ is the diagonal matrix with l-th diagonal entry equal to G_{ll}.
It follows that

$$g_N(z) = g_{\mu_{A_N}}(z-\sigma^2 g_N(z)) + \frac{1}{N}\hat{L}_N(z) + O(\frac{1}{N^2}),$$

where

$$\hat{L}_N(z) = \frac{\kappa_4}{2N^2}\sum_{i,j,k,l}U^*_{ik}U_{kj}\frac{1}{z-\sigma^2 g_N(z)-\gamma_k}\mathbb{E}[G_{ji}G_{ii}G^2_{ll}]. \tag{3.8}$$

It is easy to see that $\hat{L}_N(z)$ is bounded by $C|\Im z|^{-5}$.

Proposition 3.2. *$\hat{L}_N$ defined by* (3.8) *can be written as*

$$\hat{L}_N(z) = L_N(z) + O(\frac{1}{N}), \text{ where } L_N(z) =$$

$$\frac{\kappa_4}{2N^2}\sum_{i,l}[(G_{A_N}(z-\sigma^2 g_N(z)))^2]_{ii}[G_{A_N}(z-\sigma^2 g_N(z))]_{ii}([G_{A_N}(z-\sigma^2 g_N(z))]_{ll})^2. \tag{3.9}$$

Proof of Proposition 3.2:
Step 1: We first show that for $1 \le a, b \le N$,

$$\mathbb{E}[G_{ab}] = [G_{A_N}(z - \sigma^2 g_N(z))]_{ab} + O(\frac{1}{N}). \tag{3.10}$$

From Lemma 3.3, for any $1 \le p, k \le N$,

$$\mathbb{E}[\tilde{G}_{pk}] = \frac{\delta_{pk}}{(z - \sigma^2 g_N(z) - \gamma_k)} + \frac{\kappa_4}{2N^2} \frac{\mathbb{E}[\tilde{A}(p,k)]}{(z - \sigma^2 g_N(z) - \gamma_k)} + O(\frac{1}{N^2}).$$

Let $1 \le a, b \le N$,

$$\begin{aligned}
\mathbb{E}[G_{ab}] &= \sum_{p,k} U^*_{ap} \mathbb{E}[\tilde{G}_{pk}] U_{kb} \\
&= \sum_k U^*_{ak} \frac{1}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} \\
&+ \frac{\kappa_4}{2N^2} \sum_{p,k} U^*_{ap} \frac{\mathbb{E}[\tilde{A}(p,k)]}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} \\
&+ O(\frac{1}{N}),
\end{aligned}$$

since $\sum_{p,k} |U^*_{ap} U_{kb}| \le N$. The first term in the right-hand side of the above equation is equal to $[G_{A_N}(z - \sigma^2 g_N(z))]_{ab}$. It remains to show that the term involving $\mathbb{E}[\tilde{A}(p,k)]$ is of order $\frac{1}{N}$. Let us consider the "worst term" in the decomposition (3.7) of $\tilde{A}(p,k)$, namely the last one.

$$\begin{aligned}
&\frac{1}{2N^2} \sum_{p,k,i,j,l} U^*_{ap} \frac{1}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} U^*_{ik} U_{pj} \mathbb{E}[G_{ji} G_{ii} G^2_{ll}] \\
&\qquad = \frac{1}{2N^2} \mathbb{E}[\sum_{k,i,l} \frac{1}{(z - \sigma^2 g_N(z) - \gamma_k)} U_{kb} U^*_{ik} G_{ai} G_{ii} G^2_{ll}] \\
&\qquad = \frac{1}{2N^2} \mathbb{E}[\sum_{i,l} (U^* G_D U)_{ib} G_{ai} G_{ii} G^2_{ll}] \\
&\qquad = \frac{1}{2N^2} \mathbb{E}[\sum_{l} (G G^{(d)} U^* G_D U)_{ab} G^2_{ll}] \le \frac{1}{2N} |\Im z|^{-5}.
\end{aligned}$$

Step 2: $\hat{L}_N$ defined by (3.8) can be written as

$$\frac{\kappa_4}{2N^2} \sum_{i,l} \mathbb{E}[(U^* G_D U G)_{ii} G_{ii} G^2_{ll}].$$

First notice the following bound (see Appendix)

$$\mathbb{E}[(U^* G_D U G)_{ii} G_{ii} G^2_{ll}] - \mathbb{E}[(U^* G_D U G)_{ii}] \mathbb{E}[G_{ii}] \mathbb{E}[G_{ll}]^2 = O(\frac{1}{N}). \tag{3.11}$$

Thus,

$$\hat{L}_N(z) = \frac{\kappa_4}{2N^2}\sum_{i,l}\mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 + O(\frac{1}{N}).$$

Now, note that $\mathbb{E}[(U^*G_DUG)_{ii}] = \mathbb{E}[(U^*G_D\tilde{G}U)_{ii}]$ and, according to Lemma 3.3,

$$\begin{aligned}\mathbb{E}[(U^*G_D\tilde{G}U)_{ii}] &= \sum_{p,k}(U^*G_D)_{ip}\mathbb{E}[\tilde{G}_{pk}]U_{ki}\\ &= (U^*G_D^2U)_{ii} + \frac{\kappa_4}{2N^2}\sum_{p,k}(U^*G_D)_{ip}\mathbb{E}[\tilde{A}(p,k)](G_DU)_{ki}\\ &\quad + \sum_{p,k}(U^*G_D)_{ip}O_{pk}(\frac{1}{N^2})U_{ki}.\end{aligned}$$

Thus
$\frac{\kappa_4}{2N^2}\sum_{i,l}\mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2$

$$= \frac{\kappa_4}{2N^2}\sum_{i,l}[(G_{A_N}(z-\sigma^2 g_N(z)))^2]_{ii}\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 \tag{3.12}$$

$$+\frac{\kappa_4^2}{4N^4}\sum_{i,l,p,k}(U^*G_D)_{ip}\mathbb{E}[\tilde{A}(p,k)](G_DU)_{ki}\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 \tag{3.13}$$

$$+\frac{1}{N^2}\sum_{i,l,p,k}(U^*G_D)_{ip}O_{pk}(\frac{1}{N^2})U_{ki}\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2. \tag{3.14}$$

The last term (3.14) can be rewritten as

$$\frac{1}{N^2}\sum_{l,p,k}(U\mathbb{E}[G^{(d)}]U^*G_D)_{kp}O_{pk}(\frac{1}{N^2})\mathbb{E}[G_{ll}]^2,$$

so that one can easily see that it is a $O(\frac{1}{N})$.
The second term (3.13) can be rewritten as
$\frac{\kappa_4^2}{4N^4}\sum_{t,l,s}\mathbb{E}[G_{ll}]^2$

$$\times\Big\{[U^*G_DU\mathbb{E}[G^{(d)}]U^*G_DUG]_{ts}[G_{ts}^3 + G_{tt}G_{st}G_{ss}]$$

$$+[U^*G_DU\mathbb{E}[G^{(d)}]U^*G_DUG]_{tt}[G_{ts}G_{st}G_{ss} + G_{tt}G_{ss}^2]\Big\},$$

which is obviously a $O(\frac{1}{N})$.
Hence, Proposition 3.2 follows by rewriting the first term (3.12) using (3.10). □

From the above computations, we can state the following :

Proposition 3.3. *For $z \in \mathbb{C}^+$, $g_N(z)$ satisfies:*

$$g_N(z) = g_{\mu_{A_N}}(z-\sigma^2 g_N(z)) + \frac{1}{N}L_N(z) + O\left(\frac{1}{N^2}\right) \tag{3.15}$$

where $L_N(z)$ is given by (3.9).

4 Estimation of $g_N - \tilde{g}_N$

Proposition 4.1. *For $z \in \mathbb{C}^+$,*

$$g_N(z) - \tilde{g}_N(z) + \frac{\tilde{E}_N(z)}{N} = O(\frac{1}{N^2}), \tag{4.1}$$

where $\tilde{E}_N(z)$ is given by

$$\tilde{E}_N(z) = \{\sigma^2 \tilde{g}_N'(z) - 1\}\tilde{L}_N(z) \tag{4.2}$$

with $\tilde{L}_N(z) =$

$$\frac{\kappa_4}{2N^2}\sum_{i,l}[(G_{A_N}(z-\sigma^2\tilde{g}_N(z)))^2]_{ii}[G_{A_N}(z-\sigma^2\tilde{g}_N(z))]_{ii}([G_{A_N}(z-\sigma^2\tilde{g}_N(z))]_{ll})^2. \tag{4.3}$$

Proof of proposition 4.1: First, we are going to prove that for $z \in \mathbb{C}^+$,

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(\frac{1}{N^2}), \tag{4.4}$$

where $E_N(z)$ is given by

$$E_N(z) = \{\sigma^2 \tilde{g}_N'(z) - 1\}L_N(z). \tag{4.5}$$

For a fixed $z \in \mathbb{C}^+$, one may write the subordination equation (2.2) :

$$\tilde{g}_N(z) = g_{\mu_{A_N}}(F_{\sigma,\mu_{A_N}}(z)) = g_{\mu_{A_N}}(z - \sigma^2\tilde{g}_N(z)),$$

and the approximative matricial subordination equation (3.15) :

$$g_N(z) = g_{\mu_{A_N}}(z - \sigma^2 g_N(z)) + \frac{1}{N}L_N(z) + O\left(\frac{1}{N^2}\right).$$

The main idea is to simplify the difference $g_N(z) - \tilde{g}_N(z)$ by introducing a complex number z' likely to satisfy

$$F_{\sigma,\mu_{A_N}}(z') = z - \sigma^2 g_N(z). \tag{4.6}$$

We know by Proposition 2.1 that $F_{\sigma,\mu_{A_N}}$ is a homeomorphism from $\mathbb{C}^+$ to $\Omega_{\sigma,\mu_{A_N}}$ whose inverse $H_{\sigma,\mu_{A_N}}$ has an analytic continuation to the whole upper half-plane $\mathbb{C}^+$. Since $z - \sigma^2 g_N(z) \in \mathbb{C}^+$, $z' \in \mathbb{C}$ is well-defined by the formula :

$$z' := H_{\sigma,\mu_{A_N}}(z - \sigma^2 g_N(z)).$$

One has

$$\begin{aligned} z' - z &= -\sigma^2(g_N(z) - g_{\mu_{A_N}}(z - \sigma^2 g_N(z))) \\ &= -\sigma^2\frac{L_N(z)}{N} + O(\frac{1}{N^2}) \\ &= O(\frac{1}{N}) \end{aligned}$$

There exists thus a polynomial P with nonnegative coefficients such that

$$|z' - z| \leq \frac{P(|\Im z|^{-1})}{N}.$$

On the one hand, if

$$\frac{P(|\Im z|^{-1})}{N} \geq \frac{|\Im z|}{2},$$

or equivalently

$$1 \leq \frac{2|\Im z|^{-1} P(|\Im z|^{-1})}{N}, \tag{4.7}$$

it is enough to prove that

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(1). \tag{4.8}$$

Indeed, if we assume that (4.7) and (4.8) hold, then there exists a polynomial Q with nonnegative coefficients such that

$$\begin{aligned} |g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N}| &\leq Q(|\Im z|^{-1}) \\ &\leq Q(|\Im z|^{-1}) \frac{2|\Im z|^{-1} P(|\Im z|^{-1})}{N} \\ &\leq Q(|\Im z|^{-1}) (\frac{2|\Im z|^{-1} P(|\Im z|^{-1})}{N})^2. \end{aligned}$$

Hence,

$$g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N} = O(\frac{1}{N^2}).$$

To prove (4.8), one can notice that both $g_N(z)$ and $\tilde{g}_N(z)$ are bounded by $\frac{1}{|\Im z|}$, and that

$$|E_N(z)| \leq \left\{ \frac{\sigma^2}{|\Im z|^2} + 1 \right\} |L_N(z)|,$$

where $L_N(z) = O(1)$.
On the other hand, if

$$\frac{P(|\Im z|^{-1})}{N} \leq \frac{|\Im z|}{2},$$

one has :

$$|\Im z' - \Im z| \leq |z' - z| \leq \frac{|\Im z|}{2}$$

which implies $\Im z' \geq \frac{\Im z}{2}$ and therefore $z' \in \mathbb{C}^+$. As a consequence of (2.3), $z - \sigma^2 g_N(z) \in \Omega_{\sigma, \mu_{A_N}}$ and (4.6) is satisfied. Thus,

$$|g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N}| \leq \frac{P(|\Im z|^{-1})}{N^2},$$

or, in other words,

$$g_N(z) - \tilde{g}_N(z') - \frac{L_N(z)}{N} = O(\frac{1}{N^2}). \tag{4.9}$$

On the other hand,

$$\begin{aligned}
\tilde{g}_N(z') - \tilde{g}_N(z) &= (z - z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)} \\
&= (z - z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z - x)^2} \\
&\quad + (z - z')^2 \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)^2}.
\end{aligned}$$

Taking into account the estimation of $z' - z$ above, one has :

$$(z - z') \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z - x)^2} = -\sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N} + O(\frac{1}{N^2})$$

and

$$(z - z')^2 \int_{\mathbb{R}} \frac{d(\mu_\sigma \boxplus \mu_{A_N})(x)}{(z' - x)(z - x)^2} = O(\frac{1}{N^2}).$$

Hence

$$\tilde{g}_N(z') - \tilde{g}_N(z) + \sigma^2 \tilde{g}'_N(z) \frac{L_N(z)}{N} = O(\frac{1}{N^2}). \tag{4.10}$$

(4.4) follows from (4.9) and (4.10) since

$$\begin{aligned}
|g_N(z) - \tilde{g}_N(z) + \frac{E_N(z)}{N}| \leq \quad & |g_N(z) - \tilde{g}_N(z') - \tfrac{L_N(z)}{N}| \\
& + |\tilde{g}_N(z') - \tilde{g}_N(z) + \sigma^2 \tilde{g}'_N(z) \tfrac{L_N(z)}{N}|.
\end{aligned}$$

Now, since $E_N(z) = O(1)$, we can deduce from (4.4) that $g_N(z) - \tilde{g}_N(z) = O(\frac{1}{N})$ and then that $E_N(z) - \hat{E}_N(z) = O(\frac{1}{N})$. (4.1) readily follows. $\square$

Remark 4.1. *By combining the estimation proved above for the difference between g_N and the Stieltjes transform of $\mu_\sigma \boxplus \mu_{A_N}$ with some classical arguments developed in [29], one can recover the almost sure convergence of the spectral distribution of M_N to the free convolution $\mu_\sigma \boxplus \nu$.*

5 Inclusion of the spectrum of M_N in a neighborhood of the support of $\mu_\sigma \boxplus \mu_{A_N}$

The purpose of this section is to prove the following Theorem 5.1.

Theorem 5.1. $\forall \epsilon > 0$,

$$\mathbb{P}(\textit{ For all large } N, \mathrm{Spect}(M_N) \subset \{x, \mathrm{dist}(x, \mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})) \leq \epsilon\}) = 1.$$

The proof still uses the ideas of [23] and [31] but, since $\mu_\sigma \boxplus \mu_{A_N}$ depends on N, we need here to apply the inverse Stieltjes tranform to functions depending on N. Therefore we give the details of the proof to convince the reader that the approach still holds.

Lemma 5.1. *For any fixed large N, $\tilde{E}_N$ defined in Proposition 4.1 is the Stieltjes transform of a compactly supported distribution Λ_N on $\mathbb{R}$ whose support is included in the support of $\mu_\sigma \boxplus \mu_{A_N}$.*

The proof relies on the following characterization already used in [31].

Theorem 5.2. *[32]*

- *Let Λ be a distribution on $\mathbb{R}$ with compact support. Define the Stieltjes transform of Λ, $l : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ by*

$$l(z) = \Lambda\left(\frac{1}{z-x}\right).$$

  Then l is analytic on $\mathbb{C}\setminus\mathbb{R}$ and has an analytic continuation to $\mathbb{C}\setminus\mathrm{supp}(\Lambda)$. Moreover

  (c_1) $l(z) \to 0$ as $|z| \to \infty$,

  (c_2) there exists a constant $C > 0$, an integer $n \in \mathbb{N}$ and a compact set $K \subset \mathbb{R}$ containing $\mathrm{supp}(\Lambda)$, such that for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$|l(z)| \leq C \max\{\mathrm{dist}(z, K)^{-n}, 1\},$$

  (c_3) for any $\phi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with compact support

$$\Lambda(\phi) = -\frac{1}{\pi} \lim_{y \to 0^+} \Im \int_{\mathbb{R}} \phi(x) l(x+iy) dx.$$

- *Conversely, if K is a compact subset of $\mathbb{R}$ and if $l : \mathbb{C} \setminus K \to \mathbb{C}$ is an analytic function satisfying (c_1) and (c_2) above, then l is the Stieltjes transform of a compactly supported distribution Λ on $\mathbb{R}$. Moreover, $\mathrm{supp}(\Lambda)$ is exactly the set of singular points of l in K.*

We use here the notations and results of Section 2. If $u \in \mathbb{R}$ is not in the support of $\mu_\sigma \boxplus \mu_{A_N}$, according to (2.5), $u - \sigma^2 \tilde{g}_N(u) = F_{\sigma, \mu_{A_N}}(u)$ belongs to $\mathbb{R} \setminus \overline{U_{\sigma, \mu_{A_N}}}$ and then cannot belong to $\mathrm{Spect}(A_N)$ since $\mathrm{Spect}(A_N) \subset U_{\sigma, \mu_{A_N}}$. Hence the singular points of $\tilde{E}_N$ are included in the support of $\mu_\sigma \boxplus \mu_{A_N}$.
Now, we are going to show that for any fixed large N, $\tilde{E}_N$ satisfies (c_1) and (c_2) of Theorem 5.2. Let $C > 0$ be such that, for all large N, $\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset [-C; C]$ and $\mathrm{supp}(\mu_{A_N}) \subset [-C; C]$.

Let $\alpha > C + \sigma$. For any $z \in \mathbb{C}$ such that $|z| > \alpha$,

$$|\sigma^2 \tilde{g}_N(z)| \leq \frac{\sigma^2}{|z| - C} \leq \frac{\sigma^2}{\alpha - C} < \frac{(\alpha - C)^2}{\alpha - C} = \alpha - C$$

and

$$|z-\sigma^2\tilde{g}_N(z)| \geq \Big||z|-|\sigma^2\tilde{g}_N(z)|\Big| > |z|-(\alpha-C) > C.$$

Thus we get that for any $z \in \mathbb{C}$ such that $|z| > \alpha$,

$$\begin{aligned}
\|G_{A_N}(z-\sigma^2\tilde{g}_N(z))\| &\leq \frac{1}{|z-\sigma^2\tilde{g}_N(z)|-C} \\
&< \frac{1}{|z|-(\alpha-C)-C} \\
&< \frac{1}{|z|-\alpha}.
\end{aligned}$$

We get readily that, for $|z| > \alpha$,

$$|\tilde{E}_N(z)| \leq \frac{\kappa_4}{2}\frac{1}{(|z|-\alpha)^5}\left(\frac{\sigma^2}{(|z|-C)^2}+1\right).$$

Then, it is clear than $|\tilde{E}_N(z)| \to 0$ when $|z| \to +\infty$ and (c_1) is satisfied.
Now we are going to prove (c_2) using the approach of [31](Lemma 5.5). Denote by $\mathcal{E}_N$ the convex envelope of the support of $\mu_\sigma \boxplus \mu_{A_N}$ and define

$$K_N := \{x \in \mathbb{R}; \mathrm{dist}(x, \mathcal{E}_N) \leq 1\}$$

and

$$D_N = \{z \in \mathbb{C}; 0 < \mathrm{dist}(z, K_N) \leq 1\}.$$

- Let $z \in D_N \cap (\mathbb{C} \setminus \mathbb{R})$ with $\Re(z) \in K_N$. We have $\mathrm{dist}(z, K_N) = |\Im z| \leq 1$. We have
$$|\tilde{E}_N(z)| \leq \frac{\kappa_4}{2}\left(\sigma^2\frac{1}{|\Im z|^2}+1\right)\frac{1}{|\Im z|^5}.$$
Noticing that $1 \leq \frac{1}{|\Im z|^2}$, we easily deduce that there exists some constant C_0 such that for any $z \in D_N \cap \mathbb{C} \setminus \mathbb{R}$ with $\Re(z) \in K_N$,
$$\begin{aligned}
|\tilde{E}_N(z)| &\leq C_0|\Im z|^{-7} \\
&\leq C_0\mathrm{dist}(z, K_N)^{-7} \\
&\leq C_0 \max(\mathrm{dist}(z, K_N)^{-7}; 1).
\end{aligned}$$

- Let $z \in D_N \cap (\mathbb{C} \setminus \mathbb{R})$ with $\Re(z) \notin K_N$. Then $\mathrm{dist}(z, \mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})) \geq 1$. Since $\tilde{E}_N$ is bounded on compact subsets of $\mathbb{C} \setminus \mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})$, we easily deduce that there exists some constant $C_1(N)$ such that for any $z \in D_N$ with $\Re(z) \notin K_N$,
$$|\tilde{E}_N(z)| \leq C_1(N) \leq C_1(N)\max(\mathrm{dist}(z, K_N)^{-7}; 1).$$

- Since $|\tilde{E}_N(z)| \to 0$ when $|z| \to +\infty$, $\tilde{E}_N$ is bounded on $\mathbb{C} \setminus \overline{D_N}$. Thus, there exists some constant $C_2(N)$ such that for any $z \in \mathbb{C} \setminus \overline{D_N}$,
$$|\tilde{E}_N(z)| \leq C_2(N) = C_2(N)\max(\mathrm{dist}(z, K_N)^{-7}; 1).$$

Hence (c_2) is satisfied with $C(N) = \max(C_0, C_1(N), C_2(N))$ and $n = 7$ and Lemma 5.1 follows from Theorem 5.2. $\Box$

Proof of Theorem 5.1: Using the inverse Stieltjes tranform, we get respectively that, for any φ_N in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ with compact support,

$$\mathbb{E}[\mathrm{tr}_N(\varphi_N(M_N))] - \int_{\mathbb{R}} \varphi_N d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\varphi_N)}{N}$$

$$= \frac{1}{\pi} \lim_{y \to 0^+} \Im \int_{\mathbb{R}} \varphi_N(x) r_N(x+iy) dx,$$

where $r_N(z) = \tilde{g}_N(z) - g_N(z) + \frac{1}{N}\tilde{E}_N(z)$ satisfies, according to Proposition 4.1, for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$|r_N(z)| \leq \frac{1}{N^2} P(|\Im z|^{-1}).$$

We refer the reader to the Appendix of [17] where it is proved using the ideas of [23] that if h is an analytic function on $\mathbb{C} \setminus \mathbb{R}$ which satisfies

$$|h(z)| \leq (|z| + K)^\alpha P(|\Im z|^{-1})$$

for some polynomial P with nonnegative coefficients and degree k and for some numbers $K \geq 0$ and $\alpha \geq 0$, then there exists a polynomial Q such that

$$\limsup_{y \to 0^+} |\int_{\mathbb{R}} \varphi_N(x) h(x+iy) dx|$$

$$\leq \int_{\mathbb{R}} \int_0^{+\infty} |(1+D)^{k+1} \varphi_N(x)| (|x| + \sqrt{2}t + K)^\alpha Q(t) \exp(-t) dt dx$$

where D stands for the derivative operator. Hence, if there exists $K > 0$ such that, for all large N, the support of φ_N is included in $[-K, K]$ and $\sup_N \sup_{x \in [-K,K]} |D^p \varphi_N(x)| = C_p < \infty$ for any $p \leq k+1$, dealing with $h(z) = N^2 r_N(z)$, we deduce that for all large N,

$$\limsup_{y \to 0^+} |\int_{\mathbb{R}} \varphi_N(x) r_N(x+iy) dx| \leq \frac{C}{N^2}$$

and then

$$\mathbb{E}[\mathrm{tr}_N(\varphi_N(M_N))] - \int_{\mathbb{R}} \varphi_N d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\varphi_N)}{N} = O(\frac{1}{N^2}). \tag{5.1}$$

Let $\rho \geq 0$ be in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that its support is included in $\{|x| \leq 1\}$ and $\int \rho(x) dx = 1$. Let $0 < \epsilon < 1$. Define

$$\rho_{\frac{\epsilon}{2}}(x) = \frac{2}{\epsilon} \rho(\frac{2x}{\epsilon}),$$

$$K_N(\epsilon) = \{x, \mathrm{dist}(x, \mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N})) \leq \epsilon\}$$

and

$$f_N(\epsilon)(x) = \int_{\mathbb{R}} \mathbf{1}_{K_N(\epsilon)}(y)\rho_{\frac{\epsilon}{2}}(x-y)dy.$$

the function $f_N(\epsilon)$ is in $\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $f_N(\epsilon) \equiv 1$ on $K_N(\frac{\epsilon}{2})$; its support is included in $K_N(2\epsilon)$. Since there exists K such that, for all large N, the support of $\mu_\sigma \boxplus \mu_{A_N}$ is included in $[-K;K]$, for all large N the support of $f_N(\epsilon)$ is included in $[-K-2;K+2]$ and for any $p>0$,

$$\sup_{x\in[-K-2;K+2]} |D^p f_N(\epsilon)(x)| \leq \sup_{x\in[-K-2;K+2]} \int_{-K-1}^{K+1} |D^p \rho_{\frac{\epsilon}{2}}(x-y)|dy \leq C_p(\epsilon).$$

Thus, according to (5.1),

$$\mathbb{E}[\mathrm{tr}_N(f_N(\epsilon)(M_N))] - \int_{\mathbb{R}} f_N(\epsilon)d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(f_N(\epsilon))}{N} = O_\epsilon(\frac{1}{N^2}) \quad (5.2)$$

and

$$\mathbb{E}[\mathrm{tr}_N((f'_N(\epsilon))^2(M_N))] - \int_{\mathbb{R}} (f'_N(\epsilon))^2 d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N((f'_N(\epsilon))^2)}{N} = O_\epsilon(\frac{1}{N^2}). \quad (5.3)$$

Moreover, following the proof of Lemma 5.6 in [31], one can show that $\Lambda_N(1) = 0$. Then, the function $\psi_N(\epsilon) \equiv 1 - f_N(\epsilon)$ also satisfies

$$\mathbb{E}[\mathrm{tr}_N(\psi_N(\epsilon)(M_N))] - \int_{\mathbb{R}} \psi_N(\epsilon)d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N(\psi_N(\epsilon))}{N} = O_\epsilon(\frac{1}{N^2}). \quad (5.4)$$

Moreover, since $\psi'_N(\epsilon) = -f'_N(\epsilon)$, it comes readily from (5.3) that

$$\mathbb{E}[\mathrm{tr}_N((\psi'_N(\epsilon))^2(M_N))] - \int_{\mathbb{R}} (\psi'_N(\epsilon))^2 d(\mu_\sigma \boxplus \mu_{A_N}) - \frac{\Lambda_N((\psi'_N(\epsilon))^2)}{N} = O_\epsilon(\frac{1}{N^2}).$$

Now, since $\psi_N(\epsilon) \equiv 0$ on the support of $\mu_\sigma \boxplus \mu_{A_N}$, we deduce that

$$\mathbb{E}[\mathrm{tr}_N(\psi_N(\epsilon)(M_N))] = O_\epsilon(\frac{1}{N^2}) \quad (5.5)$$

and

$$\mathbb{E}[\mathrm{tr}_N((\psi'_N(\epsilon))^2(M_N))] = O_\epsilon(\frac{1}{N^2}). \quad (5.6)$$

By Lemma 9.1 (sticking to the proof of Proposition 4.7 in [23]), we have

$$\mathbf{V}[\mathrm{tr}_N(\psi_N(\epsilon)(M_N))] \leq \frac{C_\epsilon}{N^2}\mathbb{E}\left[\mathrm{tr}_N\{(\psi'_N(\epsilon)(M_N))^2\}\right].$$

Hence, using (5.6), one can deduce that

$$\mathbf{V}[\mathrm{tr}_N(\psi_N(\epsilon)(M_N))] = O_\epsilon(\frac{1}{N^4}). \quad (5.7)$$

Set

$$Z_{N,\epsilon} := \mathrm{tr}_N(\psi_N(\epsilon)(M_N))$$

and

$$\Omega_{N,\epsilon} = \{Z_{N,\epsilon} > N^{-\frac{4}{3}}\}.$$

From (5.5) and (5.7), we deduce that

$$\mathbb{E}\{|Z_{N,\epsilon}|^2\} = O_\epsilon(\frac{1}{N^4}).$$

Hence

$$P(\Omega_{N,\epsilon}) \leq N^{\frac{8}{3}}\mathbb{E}\{|Z_{N,\epsilon}|^2\} = O_\epsilon(\frac{1}{N^{\frac{4}{3}}}).$$

By Borel-Cantelli lemma, we deduce that, almost surely for all large N, $Z_{N,\epsilon} \leq N^{-\frac{4}{3}}$. Since $Z_{N,\epsilon} \geq \mathbf{1}_{\mathbb{R}\setminus K_N(2\epsilon)}$, it follows that, almost surely for all large N, the number of eigenvalues of M_N which are in $\mathbb{R} \setminus K_N(2\epsilon)$ is lower than $N^{-\frac{1}{3}}$ and thus obviously has to be equal to zero. The proof of Theorem 5.1 is complete.□

6 Study of $\mu_\sigma \boxplus \mu_{A_N}$

The aim of this section is to show the following inclusion of the support of $\mu_\sigma \boxplus \mu_{A_N}$ (see Theorem 6.1 below). To this aim, we will use the notations and results of Section 2. We define

$$\Theta = \{\theta_j,\, 1 \leq j \leq J\} \quad \text{and} \quad \Theta_{\sigma,\nu} = \Theta \cap (\mathbb{R} \setminus \overline{U_{\sigma,\nu}}). \tag{6.1}$$

Furthermore, for all $\theta_j \in \Theta_{\sigma,\nu}$, we set

$$\rho_{\theta_j} := H_{\sigma,\nu}(\theta_j) = \theta_j + \sigma^2 g_\nu(\theta_j) \tag{6.2}$$

which is outside the support of $\mu_\sigma \boxplus \nu$ according to (2.5), and we define

$$K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) := \mathrm{supp}(\mu_\sigma \boxplus \nu) \bigcup \left\{\rho_{\theta_j},\, \theta_j \in \Theta_{\sigma,\nu}\right\}. \tag{6.3}$$

Theorem 6.1. *For any $\epsilon > 0$,*

$$\mathrm{supp}(\mu_\sigma \boxplus \mu_{A_N}) \subset K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) + (-\epsilon, \epsilon),$$

when N is large enough.

Let us decompose μ_{A_N} as

$$\mu_{A_N} = \hat{\mu}_{\beta,N} + \hat{\mu}_{\Theta,N},$$

$$\text{where } \hat{\mu}_{\beta,N} = \frac{1}{N}\sum_{j=1}^{N-r} \delta_{\beta_j(N)} \quad \text{and} \quad \hat{\mu}_{\Theta,N} = \frac{1}{N}\sum_{j=1}^{J} k_j \delta_{\theta_j}.$$

In the following, we will denote by $D(x, \delta)$ the open disk centered on x and with radius δ. We begin with a trivial technical lemma we will need in the following.

Lemma 6.1. *Let $\mathcal{K}$ be a compact set included in $\mathbb{R}\backslash\text{supp}(\nu)$. Then $g'_{\hat{\mu}_{\beta,N}}$ (which is well defined on $\mathcal{K}$ for large N) converges to g'_ν uniformly on $\mathcal{K}$.*

Proof of Lemma 6.1: We first prove that for all $u \in \mathcal{K}$,

$$- g'_{\hat{\mu}_{\beta,N}}(u) = \frac{1}{N}\sum_{j=1}^{N-r}\frac{1}{(u-\beta_j)^2} \underset{N\to+\infty}{\longrightarrow} \int \frac{d\nu(x)}{(u-x)^2} = -g'_\nu(u). \qquad (6.4)$$

Let $\epsilon > 0$ be such that $\text{dist}(\mathcal{K}, \text{supp}(\nu)) \geq \epsilon$. For all $u \in \mathcal{K}$, let h_u be a bounded continuous function defined on $\mathbb{R}$ which coincides with $f_u(x) = 1/(u-x)^2$ on $\text{supp}(\nu) + [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$. As $\max_{1\leq j\leq N-r} \text{dist}(\beta_j(N), \text{supp}(\nu))$ tends to zero as $N \to \infty$, one can find N_0 such that, for all $N \geq N_0$, $\beta_j(N) \in \text{supp}(\nu) + [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$ for all $1 \leq j \leq N-r$. Since the sequence of measures $\hat{\mu}_{\beta,N}$ weakly converges to ν, (6.4) follows, observing that $-g'_{\hat{\mu}_{\beta,N}}(u) = \int h_u(x)\, d\hat{\mu}_{\beta,N}(x)$ and $-g'_\nu(u) = \int h_u(x) d\nu(x)$.
The uniform convergence follows from Montel's theorem, since $g'_{\hat{\mu}_{\beta,N}}$ and g'_ν are analytic on $D = \{z \in \mathbb{C}, \text{dist}(z, \text{supp}(\nu)) > \frac{\epsilon}{2}\}$ and uniformly bounded on D by $\frac{4}{\epsilon^2}$ for $N \geq N_0$. $\square$

We are now in position to give the proof of Theorem 6.1. We recall that, from (2.5),

$$\mathbb{R} \setminus \text{supp}(\mu_\sigma \boxplus \mu_{A_N}) = H_{\sigma,\mu_{A_N}}(\mathbb{R} \setminus \overline{U_{\sigma,\mu_{A_N}}}). \qquad (6.5)$$

In the proofs, we will write for simplicity U_N, H_N and F_N instead of $U_{\sigma,\mu_{A_N}}$, $H_{\sigma,\mu_{A_N}}$ and $F_{\sigma,\mu_{A_N}}$ respectively.
The main step of the proof consists in observing the following inclusion of the open set $U_{\sigma,\mu_{A_N}}$.

Lemma 6.2. *For any $\epsilon' > 0$,*

$$U_{\sigma,\mu_{A_N}} \subset \{u, \text{dist}(u, \overline{U_{\sigma,\nu}}) < \epsilon'\} \cup \{u, \text{dist}(u, \Theta_{\sigma,\nu}) < \epsilon'\}, \qquad (6.6)$$

for all large N (since the compact sets $\overline{U_{\sigma,\nu}}$ and $\Theta_{\sigma,\nu}$ are disjoint, the previous union is disjoint once ϵ' is small enough).

Proof of Lemma 6.2: Define

$$\mathcal{F}_{\epsilon'} = \{u, \text{dist}(u, \overline{U_{\sigma,\nu}}) \geq \epsilon'\} \cap \{u, \text{dist}(u, \Theta_{\sigma,\nu}) \geq \epsilon'\}.$$

We shall show that for all large N, $\mathcal{F}_{\epsilon'} \subset \mathbb{R} \setminus \overline{U_N}$.
Since $\max_{1\leq j\leq N-r} \text{dist}(\beta_j(N), \text{supp}(\nu)) \to 0$ when N goes to infinity, there exists N_0 such that for all $N \geq N_0$, the $\beta_j(N)$'s are in $\text{supp}(\nu) + (-\epsilon', \epsilon')$. Since $\text{supp}(\nu) \subset \overline{U_{\sigma,\nu}}$, it is clear that for all $N \geq N_0$, $\mathcal{F}_{\epsilon'}$ is included in $\mathbb{R} \setminus \text{Spect}A_N$. Moreover, one can readily observe that if u satisfies $\text{dist}(u, \text{supp}(\nu) + (-\epsilon', \epsilon')) \geq \sigma$ and $\text{dist}(u, \Theta) \geq \sigma$ then, for all $N \geq N_0$, $-g'_{\mu_{A_N}}(u) \leq \frac{1}{\sigma^2}$. This implies that, for all $N \geq N_0$, the open set U_N is included in the compact set

$$\mathcal{F}'_{\epsilon'} = \{u, \text{dist}(u, \text{supp}(\nu) + (-\epsilon', \epsilon')) \leq \sigma\} \cup \{u, \text{dist}(u, \Theta) \leq \sigma\}.$$

Hence, it is sufficient to show that for N large enough, the compact set $\mathcal{K}_{\epsilon'} := \mathcal{F}_{\epsilon'} \cap \mathcal{F}'_{\epsilon'}$ is contained in $\mathbb{R} \setminus \overline{U_N}$.
As ν is compactly supported, the function $u \mapsto -g'_\nu(u) = \int_{\mathbb{R}} d\nu(x)/(u-x)^2$ is continuous on $\mathbb{R} \setminus \operatorname{supp}(\nu)$. Hence it reaches its bounds on the compact set $\mathcal{K}_{\epsilon'}$ (which is obviously included in $\mathbb{R} \setminus \overline{U_{\sigma,\nu}}$) so that there exists $\alpha > 0$ such that $-g'_\nu(u) \leq \frac{1}{\sigma^2} - 2\alpha$ for any u in $\mathcal{K}_{\epsilon'}$.
According to Lemma 6.1, there exists N_0 such that for all $N \geq N_0$ and for all u in $\mathcal{K}_{\epsilon'}$,

$$|g'_{\hat{\mu}_{\beta,N}}(u) - g'_\nu(u)| \leq \frac{3\alpha}{4}. \tag{6.7}$$

At last, one can notice that N_0 may be chosen large enough so that

$$\forall N \geq N_0, \quad -g'_{\hat{\mu}_{\Theta,N}}(u) = \frac{1}{N} \sum_{j=1}^{J} \frac{k_j}{(u-\theta_j)^2} \leq \frac{\alpha}{4}. \tag{6.8}$$

This is just because for all $u \in \mathcal{F}_{\epsilon'}$, one has that: $-g'_{\hat{\mu}_{\Theta,N}}(u) \leq \frac{r}{N\epsilon'^2}$ which converges uniformly on $\mathcal{K}'_{\epsilon'}$ to 0 as N goes to infinity.
Combining all the preceding gives that, on $\mathcal{K}_{\epsilon'}$, the function $-g'_{\mu_{A_N}}$ is bounded from above by $\frac{1}{\sigma^2} - \alpha$. This implies that $\mathcal{K}_{\epsilon'}$ is included in $\mathbb{R} \setminus \overline{U_{\sigma,\mu_{A_N}}}$ which is what we wanted to show. $\square$

Now we shall establish the following inclusion.

Lemma 6.3. *For all $\epsilon > 0$, for all $\epsilon' > 0$ small enough,*

$$\mathbb{R} \setminus (K_\sigma(\theta_1, \ldots, \theta_J) + [-\epsilon, \epsilon]) \subset H_N\left(\{u, \operatorname{dist}(u, \Theta_{\sigma,\nu} \cup \overline{U_{\sigma,\nu}}) > \epsilon'\}\right), \tag{6.9}$$

when N is large enough.

Combined with Lemma 6.2, this result leads to Theorem 6.1.

Proof of Lemma 6.3: According to (2.5), (2.7) and Remark 2.1, we have that

$$\mathbb{R} \setminus \operatorname{supp}(\mu_\sigma \boxplus \nu) =$$

$$\left]-\infty, H_{\sigma,\nu}(s_m)\right[\bigcup \left(\bigcup_{l=m}^{2} \left]H_{\sigma,\nu}(t_l), H_{\sigma,\nu}(s_{l-1})\right[\right) \bigcup \left]H_{\sigma,\nu}(t_1), +\infty\right[$$

i.e.

$$\operatorname{supp}(\mu_\sigma \boxplus \nu) = \bigcup_{l=m}^{1} \Big[H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l)\Big]. \tag{6.10}$$

Note that there exists some finite integer q such that, for ϵ small enough, $\mathbb{R} \setminus (K_\sigma(\theta_1, \ldots, \theta_J) + [-\epsilon, \epsilon])$ is the following disjoint union of intervals

$$]-\infty, h_0[\bigcup_{i=1,\ldots,q}]k_i, h_i[\cup]k_{q+1}, +\infty[,$$

where $h_i = H_{\sigma,\nu}(s_{p_i})-\epsilon$ and $k_{i+1} = H_{\sigma,\nu}(t_{p_i})+\epsilon$ for some p_i or $h_i = H_{\sigma,\nu}(\theta_{j_i})-\epsilon$ and $k_{i+1} = H_{\sigma,\nu}(\theta_{j_i}) + \epsilon$ for some θ_{j_i} in $\Theta_{\sigma,\nu}$.
For such an $\epsilon > 0$, since $H_{\sigma,\nu}$ coincides on $\mathbb{R} \setminus U_{\sigma,\nu}$ with the homeomorphism $\Psi_{\sigma,\nu}$ defined in Theorem 2.1, we can deduce in particular that $H_{\sigma,\nu}$ is right-continuous (resp. left-continuous) at each t_l (resp. s_l) for $1 \leq l \leq m$, and $H_{\sigma,\nu}$ is continuous at each θ_i in $\Theta_{\sigma,\nu}$. Thus, there exists $\epsilon' > 0$ such that: for all $1 \leq l \leq m$,

$$H_{\sigma,\nu}(s_l - \epsilon') \geq H_{\sigma,\nu}(s_l) - \frac{\epsilon}{2} \quad \text{and} \quad H_{\sigma,\nu}(t_l + \epsilon') \leq H_{\sigma,\nu}(t_l) + \frac{\epsilon}{2} \tag{6.11}$$

and for all θ_j in $\Theta_{\sigma,\nu}$,

$$H_{\sigma,\nu}(\theta_j - \epsilon') \geq H_{\sigma,\nu}(\theta_j) - \frac{\epsilon}{2} \quad \text{and} \quad H_{\sigma,\nu}(\theta_j + \epsilon') \leq H_{\sigma,\nu}(\theta_j) + \frac{\epsilon}{2}. \tag{6.12}$$

Now H_N being increasing on $\mathbb{R} \setminus \overline{U_N}$, for N large enough, the image by H_N of

$$\{u,\, d(u, \Theta_{\sigma,\nu}) > \epsilon'\} \cap \{u,\, d(u, \overline{U_{\sigma,\nu}}) > \epsilon'\} \subseteq \mathbb{R} \setminus \overline{U_N}$$

is the following disjoint union of intervals

$$]-\infty, h_0(N)[\bigcup_{i=1,\ldots,q}]k_i(N), h_i(N)[\cup]k_{q+1}(N), +\infty[,$$

where $h_i(N) = H_N(s_{p_i} - \epsilon')$ and $k_{i+1}(N) = H_N(t_{p_i} + \epsilon')$ or $h_i(N) = H_N(\theta_{j_i} - \epsilon')$ and $k_{i+1}(N) = H_N(\theta_{j_i} + \epsilon')$.
One can see that it only remains to state that for all large N: $\forall 1 \leq l \leq m$,

$$H_N(s_l - \epsilon') \geq H_{\sigma,\nu}(s_l) - \epsilon \quad \text{and} \quad H_N(t_l + \epsilon') \leq H_{\sigma,\nu}(t_l) + \epsilon. \tag{6.13}$$

$$H_N(\theta_i - \epsilon') \geq H_{\sigma,\nu}(\theta_i) - \epsilon \quad \text{and} \quad H_N(\theta_i + \epsilon') \leq H_{\sigma,\nu}(\theta_i) + \epsilon. \tag{6.14}$$

Moreover, as μ_{A_N} weakly converges to ν, it is not hard to see that for all $1 \leq l \leq m$, and all θ_i in $\Theta_{\sigma,\nu}$, $H_N(s_l - \epsilon')$, $H_N(t_l + \epsilon')$, $H_N(\theta_i - \epsilon')$ and $H_N(\theta_i + \epsilon')$ converge as $N \to \infty$ to $H_{\sigma,\nu}(s_l - \epsilon')$, $H_{\sigma,\nu}(t_l + \epsilon')$, $H_{\sigma,\nu}(\theta_i - \epsilon')$ and $H_{\sigma,\nu}(\theta_i + \epsilon')$ respectively. So, there exists N_0 such that for all $N \geq N_0$: $H_N(s_l - \epsilon') \geq H_{\sigma,\nu}(s_l - \epsilon') - \frac{\epsilon}{2}$ and $H_N(t_l + \epsilon') \leq H_{\sigma,\nu}(t_l + \epsilon') + \frac{\epsilon}{2}$ as well as $H_N(\theta_i - \epsilon') \geq H_{\sigma,\nu}(\theta_i - \epsilon') - \frac{\epsilon}{2}$ and $H_N(\theta_i + \epsilon') \leq H_{\sigma,\nu}(\theta_i + \epsilon') + \frac{\epsilon}{2}$. We can then deduce (6.13) and (6.14) from (6.11) and (6.12). $\square$

7 Exact separation of eigenvalues

Before stating the fundamental exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, we need a preliminary lemma (see Lemma 7.1 below).
From Section 2, we readily deduce the following

Proposition 7.1.

$$\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) = \{x \in \mathbb{R}, F_{\sigma,\nu}(x) \in \mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}\}$$

and $F_{\sigma,\nu}$ is a homeomorphism from $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$ onto $\mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}$ with inverse $H_{\sigma,\nu}$.

Remark 7.1. *: For all $\hat{\sigma} < \sigma$, $\mathbb{R} \setminus \overline{U_{\sigma,\nu}} \subset \mathbb{R} \setminus \overline{U_{\hat{\sigma},\nu}}$ so that it makes sense to consider the following composition of homeomorphism*

$$H_{\hat{\sigma},\nu} \circ F_{\sigma,\nu} : \mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J) \to H_{\hat{\sigma},\nu}(\mathbb{R} \setminus \{\overline{U_{\sigma,\nu}} \cup \Theta\}) \subset \mathbb{R} \setminus K_{\hat{\sigma},\nu}(\theta_1, \ldots, \theta_J),$$

which is stricly increasing on each connected component of $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$.

Lemma 7.1. *Let $[a, b]$ be a compact set contained in $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \ldots, \theta_J)$. Then,*

(i) For all large N, $[F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)] \subset \mathbb{R} \setminus \mathrm{Spect}(A_N)$.

(ii) For all $0 < \hat{\sigma} < \sigma$, the interval $[H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a)), H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b))]$ is contained in $\mathbb{R} \setminus K_{\hat{\sigma},\nu}(\theta_1, \ldots, \theta_J)$ and $H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b)) - H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a)) \geq b - a$.

Proof of Lemma 7.1: For simplicity, we define $K^{\epsilon}_{\sigma,J} = K_{\sigma}(\theta_1, \ldots, \theta_J) + [-\epsilon, \epsilon]$. As $[a, b]$ is a compact set, there exist $\epsilon > 0$ and $\alpha > 0$ such that

$$[a - \alpha, b + \alpha] \subset \mathbb{R} \setminus K^{\epsilon}_{\sigma,J} \quad \text{and} \quad \mathrm{dist}([a - \alpha, b + \alpha]; K^{\epsilon}_{\sigma,J}) \geq \alpha.$$

As before, we let $\tilde{\mu}_N = \mu_\sigma \boxplus \mu_{A_N}$. According to Theorem 6.1, there exists some N_0 such that for all $N \geq N_0$, $\mathrm{supp}(\tilde{\mu}_N)$ is contained in $K^{\epsilon}_{\sigma,J}$. Thus, using (2.5) and since F_N is continuous strictly increasing on $[a - \alpha, b + \alpha]$, we have

$$\forall N \geq N_0, \quad [F_N(a - \alpha), F_N(b + \alpha)] \subset \mathbb{R} \setminus \overline{U}_N \subset \mathbb{R} \setminus \mathrm{Spect}(A_N). \tag{7.1}$$

As $F_{\sigma,\nu}$ is strictly increasing on the compact set $[a - \alpha, b + \alpha]$ ($\mathrm{supp}(\mu_\sigma \boxplus \nu) \subset K^{\epsilon}_{\sigma,J}$), one can consider $\delta > 0$ such that

$$F_{\sigma,\nu}(a - \alpha) \leq F_{\sigma,\nu}(a) - \delta \quad \text{and} \quad F_{\sigma,\nu}(b + \alpha) \geq F_{\sigma,\nu}(b) + \delta. \tag{7.2}$$

Now, the weak convergence of the probability measures $\tilde{\mu}_N$ to $\mu_\sigma \boxplus \nu$ will lead to the result, recalling from the definition of the subordination functions that for all $x \in [a - \alpha, b + \alpha]$: $F_{\sigma,\nu}(x) = x - \sigma^2 g_{\mu_\sigma \boxplus \nu}(x)$ and $F_N(x) = x - \sigma^2 g_{\tilde{\mu}_N}(x)$ (at least for all $N \geq N_0$). Indeed, observing that for any x in $[a - \alpha, b + \alpha]$, the map $h : t \mapsto \frac{1}{x-t}$ is bounded on $K^{\epsilon}_{\sigma,J}$, one readily gets the simple convergence of $g_{\tilde{\mu}_N}$ to $g_{\mu_\sigma \boxplus \nu}$ as well as the one of the corresponding subordination functions, by considering a bounded continuous function which coincides with h on $K^{\epsilon}_{\sigma,J}$. We then deduce that there exists $N_0' \geq N_0$ such that, for all $N \geq N_0'$,

$$F_N(a - \alpha) \leq F_{\sigma,\nu}(a - \alpha) + \delta \quad \text{and} \quad F_N(b + \alpha) \geq F_{\sigma,\nu}(b + \alpha) - \delta. \tag{7.3}$$

Combining (7.1), (7.2) and (7.3) proves that the inclusion of point (i) holds true for all $N \geq N_0'$.

The first part of (ii) is obvious from Remark 7.1. The second part mainly follows from the fact that $F_{\sigma,\nu}$ is strictly increasing on $\mathbb{R} \setminus \mathrm{supp}(\mu_\sigma \boxplus \nu)$. More precisely, if we set $a' = H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(a))$ and $b' = H_{\hat{\sigma},\nu}(F_{\sigma,\nu}(b))$, then

$$\begin{aligned} b' - a' &= F_{\sigma,\nu}(b) - F_{\sigma,\nu}(a) + \hat{\sigma}^2\big(g_\nu(F_{\sigma,\nu}(b)) - g_\nu(F_{\sigma,\nu}(a))\big) \\ &\geq F_{\sigma,\nu}(b) - F_{\sigma,\nu}(a) + \sigma^2\big(g_\nu(F_{\sigma,\nu}(b)) - g_\nu(F_{\sigma,\nu}(a))\big) \\ &\geq H_{\sigma,\nu}(F_{\sigma,\nu}(b)) - H_{\sigma,\nu}(F_{\sigma,\nu}(a)) = b - a \end{aligned}$$

since $F_{\sigma,\nu}(a) < F_{\sigma,\nu}(b)$ and then $g_\nu(F_{\sigma,\nu}(b)) - g_\nu(F_{\sigma,\nu}(a)) < 0$. $\Box$

The exact separation result involving the subordination function related to the free convolution of μ_σ and ν can now be stated. Let $[a, b]$ be a compact interval contained in $\mathbb{R} \setminus K_{\sigma,\nu}(\theta_1, \dots, \theta_J)$. By Theorems 5.1 and 6.1, almost surely for all large N, $[a, b]$ is outside the spectrum of M_N. Moreover, from Lemma 7.1 (i), it corresponds an interval $I = [a', b']$ outside the spectrum of A_N for all large N i.e., with the convention that $\lambda_0(M_N) = \lambda_0(A_N) = +\infty$ and $\lambda_{N+1}(M_N) = \lambda_{N+1}(A_N) = -\infty$, there is $i_N \in \{0, \dots, N\}$ such that

$$\lambda_{i_N+1}(A_N) < F_{\sigma,\nu}(a) := a' \quad \text{and} \quad \lambda_{i_N}(A_N) > F_{\sigma,\nu}(b) := b'. \tag{7.4}$$

The numbers a and a' (resp. b and b') are linked as follows:

$$a = \rho_{a'} := H_{\sigma,\nu}(a') = a' + \sigma^2 g_\nu(a'),$$

$$b = \rho_{b'} := H_{\sigma,\nu}(b') = b' + \sigma^2 g_\nu(b').$$

We claim that $[a, b]$ splits the spectrum of M_N exactly as I splits the spectrum of A_N. In other words,

Theorem 7.1. *With i_N satisfying* (7.4), *one has*

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \ \text{ and } \ \lambda_{i_N}(M_N) > b, \text{ for all large } N] = 1. \tag{7.5}$$

The proof closely follows the proof of Theorem 4.5 in [18] by introducing in a fit way the subordination functions or their inverses. For the reader's convenience, we rewrite the whole proof. The key idea is to introduce a continuum of matrices $M_N^{(k)}$ interpolating from M_N to A_N:

$$M_N^{(k)} := \frac{\sigma_k}{\sigma} \frac{W_N}{\sqrt{N}} + A_N,$$

where

$$\sigma_k^2 = \sigma^2\Big(\frac{1}{1 + kC_{a,b}}\Big),$$

and $C_{a,b}$ being a positive constant which has to be chosen small enough to ensure that the matrices $M_N^{(k)}$ and $M_N^{(k+1)}$ are close enough to each other. More precisely, $C_{a,b}$ is chosen such that

$$\max\Big(\sigma^2 C_{a,b}|g_{\mu_\sigma \boxplus \nu}(a)|; \sigma^2 C_{a,b}|g_{\mu_\sigma \boxplus \nu}(b)|; 3\sigma C_{a,b}\Big) < \frac{b-a}{4}. \tag{7.6}$$

In particular, $\sigma_0 = \sigma$ and $\sigma_k \to 0$ when k goes to infinity.
We first prove that the intervals $[H_{\sigma_k,\nu}(F_{\sigma,\nu}(a)), H_{\sigma_k,\nu}(F_{\sigma,\nu}(b))]$ split respectively the spectrum of $M_N^{(k)}$ in exactly the same way. Moreover, we also prove that for k large enough, the interval $[H_{\sigma_k,\nu}(F_{\sigma,\nu}(a)), H_{\sigma_k,\nu}(F_{\sigma,\nu}(b))]$ splits the spectrum of $M_N^{(k)}$ as $[F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)]$ splits the spectrum of A_N, this means roughly that we extend the first statement to $k = \infty$ and the result follows.

As in [18], this proof is inspired by the work [5] and mainly relies on results on eigenvalues of the rescaled Wigner matrix X_N combined with the following classical result (due to Weyl).

Lemma 7.2. *(cf. Theorem 4.3.7 of [24]) Let B and C be two $N \times N$ Hermitian matrices. For any pair of integers j, k such that $1 \le j, k \le N$ and $j + k \le N + 1$, we have*

$$\lambda_{j+k-1}(B + C) \le \lambda_j(B) + \lambda_k(C).$$

For any pair of integers j, k such that $1 \le j, k \le N$ and $j + k \ge N + 1$, we have

$$\lambda_j(B) + \lambda_k(C) \le \lambda_{j+k-N}(B + C).$$

Proof of Theorem 7.1: Given $k \ge 0$, define

$$a_k = H_{\sigma_k,\nu}(F_{\sigma,\nu}(a)) \,\text{and}\, b_k = H_{\sigma_k,\nu}(F_{\sigma,\nu}(b)).$$

Remark 7.2. *Note that in [18] where $\nu = \delta_0$, we considered $a_k = z_{\sigma_k}(g_\sigma(a))$ where g_σ denoted the Stieltjes transform of μ_σ and z_{σ_k} the inverse of g_{σ_k}. Actually, when $\nu = \delta_0$, then $H_{\sigma_k,\nu}(z) = z + \sigma_k^2/z = z_{\sigma_k}(1/z)$ and $F_{\sigma,\nu} = 1/g_\sigma$ so that $z_{\sigma_k}(g_\sigma) = H_{\sigma_k,\nu}(F_{\sigma,\nu})$. This very interpretation of the composition $z_{\sigma_k} \circ g_\sigma$ in terms of subordination function allows us to extend the result of exact separation to non-finite rank perturbations.*

The last point of (ii) in Lemma 7.1 yields $b_k - a_k \ge b - a$. Moreover

$$\begin{aligned} a_{k+1} - a_k &= (\sigma_{k+1}^2 - \sigma_k^2) g_{\mu_\sigma \boxplus \nu}(a) \\ &= -C_{a,b} \frac{\sigma^2}{(1 + kC_{a,b})(1 + (k+1)C_{a,b})} g_{\mu_\sigma \boxplus \nu}(a), \end{aligned}$$

so that $|a_{k+1} - a_k| \le \sigma^2 C_{a,b} |g_{\mu_\sigma \boxplus \nu}(a)|$. Similarly $|b_{k+1} - b_k| \le \sigma^2 C_{a,b} |g_{\mu_\sigma \boxplus \nu}(b)|$. Hence, we deduce from (7.6) that

$$|a_{k+1} - a_k| < \frac{b - a}{4} \quad \text{and} \quad |b_{k+1} - b_k| < \frac{b - a}{4}. \tag{7.7}$$

Now, we shall show by induction on k that, with probability 1, for large N, the $M_N^{(k)}$ have respectively the same amount of eigenvalues to the left sides of the interval $[a_k, b_k]$. For all $k \ge 0$, set

$$\mathrm{E}_k = \{\text{no eigenvalues of } M_N^{(k)} \text{ in } [a_k, b_k], \text{ for all large } N\}.$$

By Lemma 7.1 (*ii*) and Theorems 5.1 and 6.1, we know that $\mathbb{P}(\mathrm{E}_k) = 1$ for all k. In particular, one has for all $\omega \in \mathrm{E}_0$ and for all large N,

$$\exists j_N(\omega) \in \{0, \dots, N\} \text{ such that } \lambda_{j_N(\omega)+1}(M_N) < a \text{ and } \lambda_{j_N(\omega)}(M_N) > b. \tag{7.8}$$

Extending the random variable j_N, by setting for instance $j_N := -1$ on the complementary of E_0, we want to show that for all k,

$$\mathbb{P}[\lambda_{j_N+1}(M_N^{(k)}) < a_k \text{ and } \lambda_{j_N}(M_N^{(k)}) > b_k, \text{ for all large } N] = 1. \tag{7.9}$$

We proceed by induction. By (7.8), this is true for $k = 0$. Now, let us assume that (7.9) holds true. Since

$$M_N^{(k+1)} = M_N^{(k)} + \left(\frac{1}{\sqrt{1 + (k+1)C_{a,b}}} - \frac{1}{\sqrt{1 + kC_{a,b}}}\right)X_N,$$

we can deduce from Lemma 7.2 that

$$\lambda_{j_N+1}(M_N^{(k+1)}) \leq \lambda_{j_N+1}(M_N^{(k)}) + (-\lambda_N(X_N))C_{a,b}.$$

Since, for N large enough, $0 < -\lambda_N(X_N) \leq 3\sigma$ almost surely, it follows using (7.6) that

$$\lambda_{j_N+1}(M_N^{(k+1)}) < a_k + \frac{b-a}{4} := \hat{a}_k \quad \text{a.s..}$$

Similarly, one can show that

$$\lambda_{j_N}(M_N^{(k+1)}) > b_k - \frac{b-a}{4} := \hat{b}_k \quad \text{a.s..}$$

Inequalities (7.7) ensure that

$$[\hat{a}_k, \hat{b}_k] \subset [a_{k+1}, b_{k+1}].$$

As $\mathbb{P}(\mathrm{E}_{k+1}) = 1$, we deduce that, with probability 1,

$$\lambda_{j_N+1}(M_N^{(k+1)}) < a_{k+1} \text{ and } \lambda_{j_N}(M_N^{(k+1)}) > b_{k+1}, \quad \text{for all large } N.$$

This completes the proof by induction of (7.9).

Now, we are going to show that there exists K large enough so that, for all $k \geq K$, there is exact separation of the eigenvalues of the matrices A_N and $M_N^{(k)}$ i.e.

$$\mathbb{P}\big[\lambda_{i_N+1}(M_N^{(k)}) < a_k \text{ and } \lambda_{i_N}(M_N^{(k)}) > b_k, \quad \text{for all large } N\big] = 1. \tag{7.10}$$

There exists $\alpha > 0$ such that $[a-\alpha; b+\alpha] \subset \mathbb{R} \backslash K_{\sigma,\nu}(\theta_1, \dots, \theta_J)$. Thus according to Lemma 7.1 (i) for all large N,

$$[F_{\sigma,\nu}(a-\alpha); F_{\sigma,\nu}(b+\alpha)] \subset \mathbb{R} \setminus \mathrm{Spect}(A_N).$$

Now, there exists $\epsilon' > 0$ such that $F_{\sigma,\nu}(a-\alpha) < F_{\sigma,\nu}(a) - \epsilon'$ and $F_{\sigma,\nu}(b+\alpha) > F_{\sigma,\nu}(b) + \epsilon'$. It follows that, for all large N,

$$\lambda_{i_N+1}(A_N) < F_{\sigma,\nu}(a) - \epsilon' \quad \text{and} \quad \lambda_{i_N}(A_N) > F_{\sigma,\nu}(b) + \epsilon'. \tag{7.11}$$

Using Lemma 7.2, (7.11) and the fact that, almost surely, for all large N,

$$0 < \max(-\lambda_N(X_N), \lambda_1(X_N)) < 3\sigma,$$

we get the following inequalities.
If $i_N < N$, for all large N,

$$\begin{aligned} \lambda_{i_N+1}(M_N^{(k)}) &\leq \lambda_{i_N+1}(A_N) + \frac{\sigma_k}{\sigma}\lambda_1(X_N) \\ &< F_{\sigma,\nu}(a) - \epsilon' + \frac{\sigma_k}{\sigma}\lambda_1(X_N) \\ &= a_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(a) + \frac{\sigma_k}{\sigma}\lambda_1(X_N) - \epsilon' \\ &< a_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(a) + 3\sigma_k - \epsilon'. \end{aligned}$$

If $i_N > 0$, for all large N,

$$\begin{aligned} \lambda_{i_N}(M_N^{(k)}) &\geq \lambda_{i_N}(A_N) + \frac{\sigma_k}{\sigma}\lambda_N(X_N) \\ &> F_{\sigma,\nu}(b) + \epsilon' + \frac{\sigma_k}{\sigma}\lambda_N(X_N) \\ &= b_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(b) + \frac{\sigma_k}{\sigma}\lambda_N(X_N) + \epsilon' \\ &> b_k - \sigma_k^2 g_{\mu_\sigma \boxplus \nu}(b) - 3\sigma_k + \epsilon'. \end{aligned}$$

As $\sigma_k \to 0$ when $k \to +\infty$, there is K large enough such that for all $k \geq K$,

$$\max(|-\sigma_k^2 g_{\mu_\sigma \boxplus \nu}(a) + 3\sigma_k|, |-\sigma_k^2 g_{\mu_\sigma \boxplus \nu}(b) - 3\sigma_k|) < \epsilon'$$

and then, almost surely, for all N large enough

$$\lambda_{i_N+1}(M_N^{(k)}) < a_k \quad \text{if } \; i_N < N, \tag{7.12}$$

$$\text{and} \quad \lambda_{i_N}(M_N^{(k)}) > b_k \quad \text{if } \; i_N > 0. \tag{7.13}$$

Since $\lambda_{N+1}(M_N^{(k)}) = -\lambda_0(M_N^{(k)}) = -\infty$, (7.12) (resp. (7.13)) is obviously satisfied if $i_N = N$ (resp. $i_N = 0$). Thus, we have established that for any $i_N \in \{0, \ldots, N\}$ satisfying (7.4), (7.10) holds for all $k \geq K$ when K is large enough. Comparing this with (7.9), we deduce that $j_N = i_N$ almot surely and

$$\mathbb{P}\big[\lambda_{i_N+1}(M_N) < a \text{ and } \lambda_{i_N}(M_N) > b, \quad \text{for all large } N\big] = 1.$$

This ends the proof of Theorem 7.1. $\square$

We readily deduce the following

Corollary 7.1. *Let $\epsilon > 0$. Let us fix u in $\Theta_{\sigma,\nu} \cup \{t_l, l = 1, \ldots, m\}$ (resp. in $\Theta_{\sigma,\nu} \cup \{s_l, l = 1, \ldots, m\}$). Let us choose $\delta > 0$ small enough so that for large N, $[u+\delta; u+2\delta]$ (resp. $[u-2\delta; u-\delta]$) is included in $(\mathbb{R}\setminus\overline{U_{\sigma,\nu}})\cap(\mathbb{R}\setminus\mathrm{Spect}(A_N))$ and for any $0 \leq \delta' \leq 2\delta$, $H_{\sigma,\nu}(u+\delta')-H_{\sigma,\nu}(u) < \epsilon$ (resp. $H_{\sigma,\nu}(u)-H_{\sigma,\nu}(u-\delta') < \epsilon$). Let $i_N = i_N(u)$ be such that*

$$\lambda_{i_N+1}(A_N) < u + \delta \ \text{ and } \ \lambda_{i_N}(A_N) > u + 2\delta$$

(resp. $\lambda_{i_N+1}(A_N) < u - 2\delta$ and $\lambda_{i_N}(A_N) > u - \delta$). Then

$$\mathbb{P}\big[\lambda_{i_N+1}(M_N) < H_{\sigma,\nu}(u) + \epsilon \ \text{ and } \ \lambda_{i_N}(M_N) > H_{\sigma,\nu}(u), \ \text{for all large } N\big] = 1.$$

(resp. $\mathbb{P}\big[\lambda_{i_N+1}(M_N) < H_{\sigma,\nu}(u)$ and $\lambda_{i_N}(M_N) > H_{\sigma,\nu}(u)-\epsilon$ for all large $N\big] = 1$.)

8 Convergence of eigenvalues

In the non-spiked case $\Theta = \emptyset$ i.e. $r = 0$, the results of Theorems 6.1 and 5.1 read as: $\forall \epsilon > 0$,

$$\mathbb{P}[\mathrm{Spect}(M_N) \subset \mathrm{supp}(\mu_\sigma \boxplus \nu) + (-\epsilon, \epsilon), \text{ for all } N \text{ large}] = 1. \tag{8.1}$$

This readily leads to the following asymptotic result for the extremal eigenvalues.

Proposition 8.1. *Assume that the deformed model M_N is without spike i.e. $r = 0$. Let $k \geq 0$ be a fixed integer.*
The first largest (resp. last smallest) eigenvalues $\lambda_{1+k}(M_N)$ (resp. $\lambda_{N-k}(M_N)$) converge almost surely to the right (resp. left) endpoint of the support of $\mu_\sigma \boxplus \nu$.

Proof of Proposition 8.1: We here only focus on the convergence of the first largest eigenvalues since the other case is similar. Recalling that $\mathrm{supp}(\mu_\sigma \boxplus \nu) = \cup_{l=m}^{1}[H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l)]$, from (8.1), one has that, for all $\epsilon > 0$,

$$\mathbb{P}[\limsup_N \lambda_1(M_N) \leq H_{\sigma,\nu}(t_1) + \epsilon] = 1.$$

But as $H_{\sigma,\nu}(t_1)$ is a boundary point of $\mathrm{supp}(\mu_\sigma \boxplus \nu)$, the number of eigenvalues of M_N falling into $[H_{\sigma,\nu}(t_1) - \epsilon, H_{\sigma,\nu}(t_1) + \epsilon]$ tends almost surely to infinity as $N \to \infty$. Thus, almost surely,

$$\liminf_N \lambda_{1+k}(M_N) \geq H_{\sigma,\nu}(t_1) - \epsilon.$$

The result then follows by letting $\epsilon \to 0$. $\square$

In the spiked case where $r \geq 1$ ($\Theta \neq \emptyset$), the spectral measure μ_{M_N} still converges almost surely to $\mu_\sigma \boxplus \nu$. We shall study the impact of the spiked eigenvalues θ_i's on the local behavior of some eigenvalues of M_N.
In particular, we shall prove that once the largest spike θ_1 is sufficiently big, the

largest eigenvalue of M_N jumps almost surely above the right endpoint $H_{\sigma,\nu}(t_1)$. Once $m \geq 2$, that is when $\mathrm{supp}(\mu_\sigma \boxplus \nu)$ has at least two connected components, we prove that there may also exist some jumps into the gap(s) of this support. This phenomenon holds for any $\theta_j \in \Theta_{\sigma,\nu}$.

For $\theta_j \notin \Theta_{\sigma,\nu}$, that is if $\theta_j \in \overline{U_{\sigma,\nu}}$, two situations may occur. To explain this, let us consider the connected component $[s_{l_j}, t_{l_j}]$ of $\overline{U_{\sigma,\nu}}$ which contains θ_j. If $\mathrm{supp}(\nu) \cap [\theta_j, t_{l_j}] = \emptyset$ (resp. $\mathrm{supp}(\nu) \cap [s_{l_j}, \theta_j] = \emptyset$) then the k_j corresponding eigenvalues of M_N converge almost surely to the corresponding boundary point $H_{\sigma,\nu}(t_{l_j})$ (resp. $H_{\sigma,\nu}(s_{l_j})$) of the support of $\mu_\sigma \boxplus \nu$. Otherwise, namely when θ_j is between two connected components of $\mathrm{supp}(\nu)$ included in $[s_{l_j}, t_{l_j}]$, the convergence occurs towards a point inside the (interior) of $\mathrm{supp}(\mu_\sigma \boxplus \nu)$.

Here is the precise formulation of our result. This is the additive analogue of the main result of [6] on the almost sure convergence of the eigenvalues generated by the spikes in a generalized spiked population model.

Theorem 8.1. *For each spiked eigenvalue θ_j, we denote by $n_{j-1}+1, \ldots, n_{j-1}+k_j$ the descending ranks of θ_j among the eigenvalues of A_N.*

1) *If $\theta_j \in \mathbb{R} \setminus \overline{U_{\sigma,\nu}}$ (i.e. $\in \Theta_{\sigma,\nu}$), the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely outside the support of $\mu_\sigma \boxplus \nu$ towards $\rho_{\theta_j} = H_{\sigma,\nu}(\theta_j)$.*

2) *If $\theta_j \in \overline{U_{\sigma,\nu}}$ then we let $[s_{l_j}, t_{l_j}]$ (with $1 \leq l_j \leq m$) be the connected component of $\overline{U_{\sigma,\nu}}$ which contains θ_j.*

   a) *If θ_j is on the right (resp. on the left) of any connected component of $\mathrm{supp}(\nu)$ which is included in $[s_{l_j}, t_{l_j}]$ then the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely to $H_{\sigma,\nu}(t_{l_j})$ (resp. $H_{\sigma,\nu}(s_{l_j})$) which is a boundary point of the support of $\mu_\sigma \boxplus \nu$.*

   b) *If θ_j is between two connected components of $\mathrm{supp}(\nu)$ which are included in $[s_{l_j}, t_{l_j}]$ then the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely to the α_j-th quantile of $\mu_\sigma \boxplus \nu$ (that is to q_{α_j} defined by $\alpha_j = (\mu_\sigma \boxplus \nu)(]-\infty, q_{\alpha_j}])$) where α_j is such that $\alpha_j = 1 - \lim_N \frac{n_{j-1}}{N} = \nu(]-\infty, \theta_j])$.*

Proof of Theorem 8.1: 1) Choosing $u = \theta_j$ in Corollary 7.1 gives, for any $\epsilon > 0$,

$$\rho_{\theta_j} - \epsilon \leq \lambda_{n_{j-1}+k_j}(M_N) \leq \cdots \leq \lambda_{n_{j-1}+1}(M_N) \leq \rho_{\theta_j} + \epsilon, \text{ for large } N \tag{8.2}$$

holds almost surely. Hence

$$\forall 1 \leq i \leq k_j, \quad \lambda_{n_{j-1}+i}(M_N) \xrightarrow{a.s.} \rho_{\theta_j}.$$

2) a) We only focus on the case where θ_j is on the right of any connected component of $\mathrm{supp}(\nu)$ which is included in $[s_{l_j}, t_{l_j}]$ since the other case may be considered with similar arguments. Let us consider the set $\{\theta_{j_0} > \ldots > \theta_{j_p}\}$ of all the θ_i's being in $[s_{l_j}, t_{l_j}]$ and on the right of any connected component of

supp(ν) which is included in $[s_{l_j}, t_{l_j}]$. Note that we have for all large N, for any $0 \leq h \leq p$,

$$n_{j_h - 1} + k_{j_h} = n_{j_h}$$

and θ_{j_0} is the largest eigenvalue of A_N which is lower than t_{l_j}. Let $\epsilon > 0$. Applying Corollary 7.1 with $u = t_{l_j}$, we get that, almost surely,

$$\lambda_{n_{j_0 - 1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \epsilon \text{ and } \lambda_{n_{j_0 - 1}}(M_N) > H_{\sigma,\nu}(t_{l_j}) \text{ for all large } N.$$

Now, almost surely, the number of eigenvalues of M_N being in $]H_{\sigma,\nu}(t_{l_j}) - \epsilon, H_{\sigma,\nu}(t_{l_j})]$ should tend to infinity when N goes to infinity. Since almost surely for all large N, $\lambda_{n_{j_0 - 1}}(M_N) > H_{\sigma,\nu}(t_{l_j})$ and $\lambda_{n_{j_0 - 1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \epsilon$, we should have

$$H_{\sigma,\nu}(t_{l_j}) - \epsilon \leq \lambda_{n_{j_p - 1}+k_{j_p}}(M_N) \leq \ldots \leq \lambda_{n_{j_0 - 1}+1}(M_N) < H_{\sigma,\nu}(t_{l_j}) + \epsilon.$$

Hence, we deduce that: $\forall 0 \leq l \leq p$ and $\forall 1 \leq i \leq k_{j_p}$, $\quad \lambda_{n_{j_p - 1}+i}(M_N) \xrightarrow{a.s.} H_{\sigma,\nu}(t_{l_j})$. The result then follows since $j \in \{j_0, \ldots, j_p\}$.
b) Let $\alpha_j = 1 - \lim_N \frac{n_{j-1}}{N} = \nu(] - \infty, \theta_j])$. Denote by Q (resp. Q_N) the distribution function of $\mu_\sigma \boxplus \nu$ (resp. of the spectral measure of M_N). Since $\mu_\sigma \boxplus \nu$ is absolutely continuous, Q is continuous on $\mathbb{R}$ and strictly increasing on each interval $[\Psi_{\sigma,\nu}(s_l), \Psi_{\sigma,\nu}(t_l)], 1 \leq l \leq m$.
From Proposition 2.3 and the hypothesis on θ_j, $\alpha_j \in]Q(\Psi_{\sigma,\nu}(s_{l_j})), Q(\Psi_{\sigma,\nu}(t_{l_j}))[$ and there exists a unique $q_j \in]\Psi_{\sigma,\nu}(s_{l_j}), \Psi_{\sigma,\nu}(t_{l_j})[$ such that $Q(q_j) = \alpha_j$. Moreover, Q is strictly increasing in a neighborhood of q_i.
Let $\epsilon > 0$. From the almost sure convergence of μ_{M_N} to $\mu_\sigma \boxplus \nu$, we deduce

$$Q_N(q_j + \epsilon) \underset{N\to\infty}{\longrightarrow} Q(q_j + \epsilon) > \alpha_j, \quad \text{a.s..}$$

From the definition of α_j, it follows that for large N, $N, N-1, \ldots, n_{j-1} + k_j, \ldots, n_{j-1} + 1$ belong to the set $\{k, \lambda_k(M_n) \leq q_j + \epsilon\}$ and thus,

$$\limsup_{N \longrightarrow \infty} \lambda_{n_{j-1}+1}(M_N) \leq q_j + \epsilon.$$

In the same way, since $Q_N(q_j - \epsilon) \longrightarrow_{N\to\infty} Q(q_j - \epsilon) < \alpha_j$,

$$\liminf_{N \longrightarrow \infty} \lambda_{n_{j-1}+k_j}(M_N) \geq q_j - \epsilon.$$

Thus, the k_j eigenvalues $(\lambda_{n_{j-1}+i}(M_N),\ 1 \leq i \leq k_j)$ converge almost surely to q_j. $\square$

9 Appendix

We present in this appendix the different estimates on the variance used throughout the paper. They rely on the Poincaré hypothesis on the distribution μ of the entries of the Wigner matrix W_N. We assume that μ satisfies a Poincaré inequality, that is there exists a positive constant C such that for any $\mathcal{C}^\infty$ function $f : \mathbb{R} \to \mathbb{C}$ such that f and f' are in $L^2(\mu)$,

$$\mathbf{V}(f) \leq C \int |f'|^2 d\mu,$$

with $\mathbf{V}(f) = \mathbb{E}(|f - \mathbb{E}(f)|^2)$.
We refer the reader to [16] for a characterization of such measures on $\mathbb{R}$. This inequality translates in the matricial case as follows:
For any matrix M, define $||M||_2 = (\mathrm{Tr}(M^*M))^{\frac{1}{2}}$ the Hilbert-Schmidt norm. Let $\Psi : (M_N(\mathbb{C})_{sa}) \to \mathbb{R}^{N^2}$ be the canonical isomorphism which maps a Hermitian matrix M to the real parts and the imaginary parts of its entries $M_{ij}, i \leq j$.

Lemma 9.1. *Let M_N be the complex Wigner Deformed matrix introduced in Section 1. For any $\mathcal{C}^\infty$ function $f : \mathbb{R}^{N^2} \to \mathbb{C}$ such that f and its gradient $\nabla(f)$ are both polynomially bounded,*

$$\mathbf{V}[f \circ \Psi(M_N)] \leq \frac{C}{N} \mathbb{E}\{\|\nabla \left[f \circ \Psi(M_N)\right] \|_2^2\}. \tag{9.1}$$

From this Lemma and the properties of the resolvent G (see Lemma 1.1), we obtain:

- $\mathbf{V}((G_N(z))_{ij}) \leq \frac{C}{N} P(|\Im z|^{-1})$
- $\mathbf{V}((G_N(z))^2_{ii}) \leq \frac{C}{N} P(|\Im z|^{-1})$
- Let H be a deterministic Hermitian matrix with norm $\|H\|$, then,

$$\mathbf{V}((HG_N(z))_{ii}) \leq \frac{C}{N} \|H\|^2 P(|\Im z|^{-1})$$

- $\mathbf{V}(\mathrm{tr}_N(G_N(z))) \leq \frac{C}{N^2} P(|\Im z|^{-1})$

where P is a polynomial. It follows that:

$$\mathbb{E}[(U^*G_DUG)_{ii}G_{ii}G_{ll}^2] \quad = \quad \mathbb{E}[(U^*G_DUG)_{ii}]\mathbb{E}[G_{ii}]\mathbb{E}[G_{ll}]^2 + \frac{1}{N} P(|\Im z|^{-1}),$$

proving (3.11).
We now prove

Lemma 9.2. *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then,*

$$|\mathbb{E}[\tilde{G}_{pk} \mathrm{tr}_N(G)] - \mathbb{E}[\tilde{G}_{pk}]\mathbb{E}[\mathrm{tr}_N(G)]| \leq \frac{P(|\Im z|^{-1})}{N^2}.$$

Proof: The cumulant expansion gives

$$z\mathbb{E}(G_{ji}) = \sigma^2 \mathbb{E}(\mathrm{tr}_N(G)G_{ji}) + \delta_{ij} + \mathbb{E}[(GA_N)_{ji}] + \frac{\kappa_4}{2N^2}\mathbb{E}[T(i,j)] + O_{ji}(\frac{1}{N^2}),$$

where

$$\begin{aligned} T(i,j) &= \frac{1}{3}\Bigg\{ \frac{1}{\sqrt{2}} \sum_{l<i} \Big(G^{(3)}_{jl}.(e_{li},e_{li},e_{li}) + \sqrt{-1} G^{(3)}_{jl}.(f_{li},f_{li},f_{li}) \Big) \\ &+ \frac{1}{\sqrt{2}} \sum_{l>i} \Big(G^{(3)}_{jl}.(e_{il},e_{il},e_{il}) - \sqrt{-1} G^{(3)}_{jl}.(f_{il},f_{il},f_{il}) \Big) \\ &+ G^{(3)}_{jl}.(E_{ii},E_{ii},E_{ii}) \Bigg\}. \end{aligned}$$

Straightforward computations give that

$$\begin{aligned} T(i,j) = \quad & \textstyle\sum_l G_{jl} G^3_{li} + \sum_l G_{ji} G_{il} G_{li} G_{ll} \\ & + \textstyle\sum_l G_{jl} G_{ii} G_{li} G_{ll} + \sum_l G_{ji} G_{ii} G^2_{ll} - 2G^3_{ii} G_{ji}. \end{aligned}$$

We now compute the sum $\sum U^*_{ik} U_{pj} \dots$ to obtain:

$$\begin{aligned} (z-\gamma_k)\mathbb{E}[\tilde{G}_{pk}] = \quad & \sigma^2 \mathbb{E}[\mathrm{tr}_N(G)\tilde{G}_{pk}] + \delta_{pk} + \tfrac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)] \\ & - \tfrac{\kappa_4}{N^2} \textstyle\sum_{i,j} U^*_{ik} U_{pj} \mathbb{E}[G^3_{ii} G_{ji}] + \sum_{i,j} U^*_{ik} U_{pj} O_{ji}(\tfrac{1}{N^2}), \end{aligned} \tag{9.2}$$

where

$$\tilde{A}(p,k) = \sum_{i,j} U^*_{ik} U_{pj} A(i,j)$$

and

$$\begin{aligned} A(i,j) = \quad & \textstyle\sum_l G_{jl} G^3_{li} + \sum_l G_{ji} G_{il} G_{li} G_{ll} \\ & + \textstyle\sum_l G_{jl} G_{ii} G_{li} G_{ll} + \sum_l G_{ji} G_{ii} G^2_{ll}. \end{aligned}$$

Since $\frac{\kappa_4}{N^2} \sum_{i,j} U^*_{ik} U_{pj} G^3_{ii} G_{ji} = \frac{\kappa_4}{N^2} (UG(G^{(d)})^3 U^*)_{pk}$, this term is obviously a $O(\frac{1}{N^2})$.

Let us verify the following bound for $\tilde{A}$:

$$\left|\frac{1}{N^2}\tilde{A}(p,k)\right| \le C \frac{|\Im z|^{-4}}{N}. \tag{9.3}$$

Such a bound for the first term in the decomposition of A can be readily deduced from (1.2). We write the computation for the fourth term in the decomposition of A, the other two terms are similar:

$$\begin{aligned} & \frac{1}{N^2} \sum_{i,j,l} U^*_{ik} U_{pj} G_{ji} G_{ii} G^2_{ll} \\ & \qquad = \frac{1}{N^2} \sum_l (UGG^{(d)}U^*)_{pk} G^2_{ll} = O(\frac{1}{N}). \end{aligned}$$

We prove now that the last term in (9.2) is of order $O(\frac{1}{N^2})$. This term is a linear combination of terms of the form:

$$\frac{\kappa_6}{N^3} \sum_{i,j,l} U^*_{ik} U_{pj} \mathbb{E}[G^{(5)}_{jl}.(v_1,\dots,v_5)],$$

where $v_u = E_{mn}$ with $(m,n) = (i,l)$ or $(m,n) = (l,i)$. The fifth derivative is a product of six G. If there are G_{il}^2 or $G_{il}G_{li}$ in the product, we can conclude thanks to Lemma 1.1. The only term without any G_{il} is

$$G_{ji}G_{ll}G_{ii}G_{ll}G_{ii}G_{ll}$$

which gives the contribution

$$\frac{1}{N^3}\sum_l (UG(G^{(d)})^2U^*)_{pk}G_{ll}^3 = O(\frac{1}{N^2}).$$

The term with one G_{il} (or G_{li}) will also give a contribution in $\frac{1}{N^2}$. Hence

$$(z-\gamma_k)\mathbb{E}[\tilde{G}_{pk}] = \sigma^2\mathbb{E}[\mathrm{tr}_N(G)\tilde{G}_{pk}] + \delta_{pk} + \frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)] + O(\frac{1}{N^2}). \quad (9.4)$$

We now apply (3.1) (or its extension (3.2)) to $\Phi(X_N) = G_{jl}G_{qq}$ and $H = E_{il}$ and take the sum in l. We obtain

$$\begin{aligned} z\mathbb{E}(G_{ji}G_{qq}) = \quad & \sigma^2\mathbb{E}(\mathrm{tr}_N(G)G_{ji}G_{qq}) + \tfrac{\sigma^2}{N}\mathbb{E}[G_{qi}(G^2)_{jq}] + \mathbb{E}[G_{qq}\delta_{ij}] \\ & +\mathbb{E}[(GA_N)_{ji}G_{qq}] + \tfrac{\kappa_4}{2N^2}\mathbb{E}[T(i,j)G_{qq}] \\ & +\tfrac{\kappa_4}{2N^2}\mathbb{E}[B(i,j,q)] + O_{j,i}(\tfrac{1}{N^2}), \end{aligned}$$

where $B(i,j,q)$ stands for all the terms coming from the third derivative of the product $(G_{jl}G_{qq})$ except $G_{qq}G_{jl}^{(3)}$. Now, we consider $\frac{1}{N}\sum_q$ of the above equalities to obtain:

$$\begin{aligned} z\mathbb{E}(G_{ji}\mathrm{tr}_N(G)) = \quad & \sigma^2\mathbb{E}(\mathrm{tr}_N(G)^2G_{ji}) + \tfrac{\sigma^2}{N^2}\mathbb{E}[(G^3)_{ji}] + \mathbb{E}[\mathrm{tr}_N(G)\delta_{ij}] \\ & +E[(GA_N)_{ji}\mathrm{tr}_N(G)] + \tfrac{\kappa_4}{2N^2}\mathbb{E}[T(i,j)\mathrm{tr}_N(G)] \\ & +\tfrac{\kappa_4}{2N^2}\tfrac{1}{N}\textstyle\sum_q \mathbb{E}[B(i,j,q)] + O_{j,i}(\tfrac{1}{N^2}). \end{aligned}$$

We now compute the sum $\sum U_{ik}^*U_{pj}\ldots$ and obtain

$$\begin{aligned} (z-\gamma_k)\mathbb{E}(\tilde{G}_{pk}\mathrm{tr}_N(G)) = \quad & \sigma^2\mathbb{E}(\mathrm{tr}_N(G)^2\tilde{G}_{pk}) + \tfrac{\sigma^2}{N^2}\mathbb{E}[(UG^3U^*)_{pk}] \\ & +\mathbb{E}[\mathrm{tr}_N(G)\delta_{pk}] + \tfrac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)\mathrm{tr}_N(G)] \\ & +\tfrac{\kappa_4}{2N^2}\tfrac{1}{N}\textstyle\sum_q \mathbb{E}[\tilde{B}(p,k,q)] + O(\tfrac{1}{N^2}), \end{aligned}$$

where

$$\tilde{B}(p,k,q) = \sum U_{ik}^*U_{pj}B(i,j,q)$$

and the terms $\frac{\kappa_4}{2N^2}\sum U_{ik}^*U_{pj}\mathbb{E}[(T(i,j) - A(i,j))\mathrm{tr}_N(G)]$ and $\sum U_{ik}^*U_{pj}O_{j,i}(\frac{1}{N^2})$ remain a $O(\frac{1}{N^2})$ by the same arguments used to handle the analogous terms in (9.2).

Now, consider the difference between the above equation and $g_N(z)\times$(9.2):

$$
\begin{aligned}
(z-\gamma_k)\mathbb{E}[(\tilde{G}_{pk}(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])] = \\
\frac{\sigma^2}{N^2}\mathbb{E}[(UG^3U^*)_{pk}] + \sigma^2\mathbb{E}[\mathrm{tr}_N(G)(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])\tilde{G}_{pk}] \\
+\frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])] \\
+\frac{\kappa_4}{2N^2}\frac{1}{N}\sum_q \mathbb{E}[\tilde{B}(p,k,q)] + O(\frac{1}{N^2})
\end{aligned}
$$

and

$$
\begin{aligned}
(z-\gamma_k-\sigma^2 g_N(z))\mathbb{E}[\tilde{G}_{pk}(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])] = \\
\sigma^2\mathbb{E}[(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])^2\tilde{G}_{pk}] + \frac{\sigma^2}{N^2}\mathbb{E}[(UG^3U^*)_{pk}] \\
+\frac{\kappa_4}{2N^2}\mathbb{E}[\tilde{A}(p,k)(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])] \\
+\frac{\kappa_4}{2N^2}\frac{1}{N}\sum_q \mathbb{E}[\tilde{B}(p,k,q)] + O(\frac{1}{N^2}).
\end{aligned}
$$

We now prove that the right-hand side of the above equation is of order $\frac{1}{N^2}$. This is obvious for the second and first term (since $\mathbf{V}(\mathrm{tr}_N(G_N(z))) = O(\frac{1}{N^2})$). Now, we have seen that

$$
\frac{1}{N^2}\tilde{A}(p,k) \le \frac{C|\Im z|^{-4}}{N}.
$$

By Cauchy-Schwarz inequality,

$$
\frac{1}{N^2}\mathbb{E}[\tilde{A}(p,k)(\mathrm{tr}_N(G)-\mathbb{E}[\mathrm{tr}_N(G)])] = O(\frac{1}{N^2}).
$$

It remains to study the last term

$$
\frac{1}{N^3}\sum_q \mathbb{E}[\tilde{B}(p,k,q)] = \frac{1}{N^3}\sum_{i,j,q} U^*_{ik}U_{pj}\mathbb{E}[B(i,j,q)].
$$

This term contains derivatives of G_{qq} of order a with *a strictly positive* ($a = 1,2,3$) applied to a 3-tuple (v_1, v_2, v_3) where $v_u = E_{il}$ or E_{li} (with a product of the derivative of order $3-a$ of G_{jl}). Thus, the index q appears in $\tilde{B}(p,k,q)$ under the form of a product $G_{qm}G_{nq}$ with $m,n \in \{i,l\}$. Thus, the sum in q will give G^2_{nm}. Moreover, the term in j in the derivative appears as G_{jm} with $m \in \{i,l\}$ and we can do the sum in j to obtain $(UG)_{pm}$. Thus, $\frac{1}{N^3}\sum_q \tilde{B}(p,k,q)$ can be written as $\frac{1}{N^3}\sum_{i,l}$ of terms of the form

$$
U^*_{ik}(G^2)_{i_1j_1}(UG)_{pj_2}G_{i_3j_3}G_{i_4j_4},
$$

where $i_r, j_r \in \{i,l\}$ and $j_2 = l$ for $a = 3$ (no derivative in G_{jl}), $j_4 = l$ for $a < 3$. As in the previous computations, either the product G^2_{il} (or $G_{il}G_{li}$) appears and we can apply Lemma 1.1 (the others terms are bounded). In the other cases, we can always perform one sum in i (or l) and obtain $\frac{1}{N^3}\sum_{l(\text{ or } i)}$ of bounded terms. Let us just give an example of terms which can be obtained (for $a = 1$):

$$U^*_{ik}(G^2)_{li}(UG)_{pl}G_{ii}G_{ll}.$$

Then,

$$\frac{1}{N^3}\sum_{i,l} U^*_{ik}(G^2)_{li}(UG)_{pl}G_{ii}G_{ll} = \frac{1}{N^3}\sum_{i} U^*_{ik}(UGG^{(d)}G^2)_{pi}G_{ii}.$$

Therefore, $\frac{1}{N^3}\sum_q \mathbb{E}[\tilde{B}(p,k,q)]$ is of order $\frac{1}{N^2}$. This proves Lemma 9.2 since $|\frac{1}{z-\gamma_k-\sigma^2 g_N(z)}| \leq |\Im z|^{-1}$. $\square$

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